DIRECTORATE OF DISTANCE EDUCATION

UNIVERSITY OF NORTH BENGAL

MASTER OF SCIENCES- MATHEMATICS

SEMESTER -IV

ABSTRACT MEASURE THEORY

DEMATH4CORE1

BLOCK-2

UNIVERSITY OF NORTH BENGAL

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FOREWORD

The Self Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours

ABSTRACT MEASURE THEORY

BLOCK-1

BLOCK-2

INTRODUCTION TO BLOCK-II

This block discusses about σ-algebra, its monotone classes, its restrictions and about Borel σ-algebra.we study about general measures, Point mass distributions, complete measures, restrictions and its uniqueness. We discusses different kinds of borel measures, outer measures and its constructions ,volume of intervals , lebesgue measure and its transformations and also about cantor set,cantor ternary set and its functions,different functions and arithmetic operations which we can perform on the measurable functions.

In this block We will be learning about the devil's staircase and seeing problems related to it.

UNIT 8 CANTOR–LEBESGUE FUNCTION

STRUCTURE

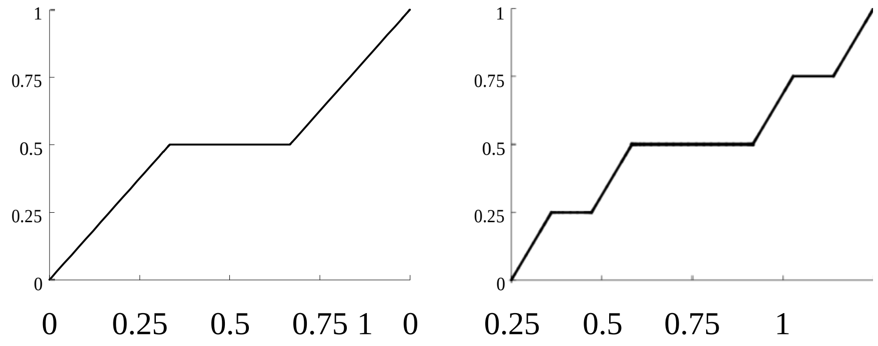
8.1 OBJECTIVE

In this chapter we are going to learn about the cantor lebesgue functions , its lemmas , its theorem's and problems related on it.We study about the completeness of a measure spaces, its definitions and see problems related to it

8.2 INTRODUCTION

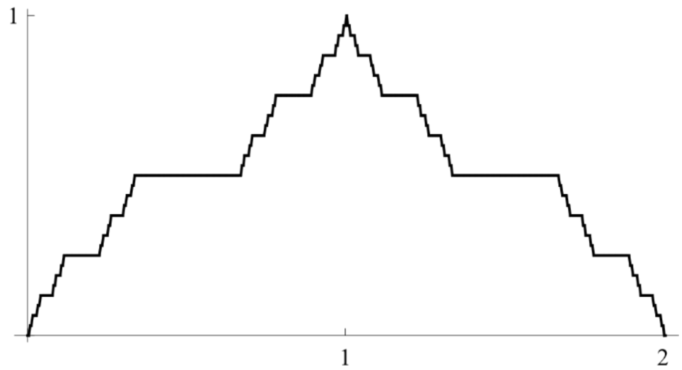
Consider the two functions ϕ_1, ϕ_2 pictured in. The function ϕ_1 takes the constant value $\frac{1}{2}$ on the interval ($\frac{1}{3}, \frac{2}{3}$) that is removed from [0,1] in the first stage of the construction of the Cantor middle-thirds set, and is linear on the remaining intervals. The function ϕ_2 takes the same constant $\frac{1}{2}$ on the interval ($\frac{1}{3}, \frac{2}{3}$) but additionally is constant with values $\frac{1}{4}$ and $\frac{3}{4}$ on the two intervals that are removed in the second stage of the construction of the Cantor set. We continue this process and define $\phi_3, \phi_4, \ldots$ in a similar way. Each function ϕk is continuous, and is constant on each of the open intervals that were removed at the kth stage of the construction of the

Cantor set. The following exercise shows that these functions converge uniformly to a continuous function.



Top left: The function ϕ_1. Top right: The function ϕ_2.

This limit function ϕ is called the *Cantor–Lebesgue function* or, more picturesquely, the ***Devil's staircase.*** If we extend ϕ to R by reflecting it about the point $x = 1$ and declaring it to be zero outside of [0,2], we obtain the continuous function ϕ .



The reflected Devil's staircase (Cantor–Lebesgue function).

The Cantor–Lebesgue function is not Lipschitz, but it does satisfy a weaker condition.

Exercise 1.57. Prove the following facts.

(a) Each function ϕ_k is monotone increasing on the interval [0, 1], and

$|\phi_{k+1}(x) - \phi_k(x)| < 2^{-k}$ for every $x \in [0, 1]$.

(b) The functions ϕ_k converge uniformly on [0, 1], and the limit function $\phi(x) = \lim_{k\to\infty} \phi_k(x)$ is continuous on [0, 1]. Moreover, ϕ is differentiable at almost every point $x \in [0, 1]$, and although ϕ is not differentiable at all points, we have $\phi'(x) = 0$ a.e. in [0,1]. ♦

This limit function ϕ is called the Cantor-Lebesgue function or, more picturesquely, the Devil's staircase. If we extend ϕ to R by reflecting it about the point x = 1 and declaring it to be zero outside of [0,2], we obtain the continuous function.

8.3 LEMMA'S AND THEOREM'S

Definition 8.2. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *Hölder continuous* on R with exponent $\alpha > 0$ if there exists a constant $C > 0$ such that

$$\forall\, x,y \in \mathbb{R}, \qquad |f(x) - f(y)| \le C\,|x - y|^{\alpha}. \qquad ♦$$

Thus Lipschitz continuity on R is Hölder continuity with exponent α = 1.

We will use the Cantor–Lebesgue function to derive some interesting insights into the behavior of measurable sets under continuous functions. First we show that a continuous function can map a set with zero measure to a set with positive measure.

Lemma 8.2. *The Cantor–Lebesgue function ϕ maps the Cantor set C, which has zero measure, to a set that has positive Lebesgue measure.*

Proof. If $x \notin C$, then x belongs to one of the open intervals removed at some stage in forming the Cantor set. Consequently $\phi(x)$ is a dyadic rational number, i.e., $\phi(x) = m/2^n$ for some integers m and n. Therefore ϕ maps the complement of the Cantor set into the set of rationals in [0,1], which is a countable set. Consequently $\phi(C)$ includes all of the irrational numbers in [0,1], so $\phi(C) = [0,1]\setminus Z$ where $Z \subseteq \mathbb{Q}$. Since Z has measure zero, it follows that $\phi(C)$ is measurable and $|\phi(C)| = 1$. ⊓⊔

Second, we show that a continuous function need not map a measurable set to a measurable set.

Lemma 8.3. *Let ϕ be the Cantor–Lebesgue function. There exists a measurable set $E \subseteq [0,1]$ such that $\phi(E)$ is not measurable.*

Proof. Let N be a nonmeasurable subset of [0,1]. By replacing N with $N\setminus\mathbb{Q}$, we may assume that N contains no rational numbers. Consequently $\phi^{-1}(N)$ is contained in the Cantor set C. Since C has zero measure, monotonicity implies that $|\phi^{-1}(N)| = 0$, so $E = \phi^{-1}(N)$ is measurable. However, since ϕ is surjective, the image of E under ϕ is N, which is not measurable.

Check your progress

1.1) Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is Hölder continuous for some exponent $\alpha > 1$, then f is constant.

8.4 COMPLETENESS OF A MEASURE SPACES

By definition, a set $E \subseteq X$ is a null set for a measure μ on X if $E \in \Sigma$ and $\mu(E) = 0$. In general, an arbitrary subset A of E need not be measurable, but if A happens to be measurable then monotonicity implies that $\mu(A) = 0$. A *complete measure* is one such that every subset A of every null set E is measurable.

Complete measures are often more convenient to work with than incomplete measures. Fortunately, if we have a incomplete measure μ in hand, there

is a way to extend μ to a larger σ-algebra $\bar{\Sigma}$ in such a way that the extended measure is complete. This new extended measure $\bar{\mu}$ is called the *completion* of μ, and its construction is given in the next exercise.

Check your progress

1.2) Let (X, Σ, μ) be a measure space, and let N be the collection of all μ-null sets in X:

$$\mathcal{N} = \{N \in \Sigma : \mu(N) = 0\}.$$

Define

$$\overline{\Sigma} = \{E \cup Z : E \in \Sigma,\ Z \subseteq N \in \mathcal{N}\},$$

and prove the following statements.

(a) $\overline{\Sigma}$ is a σ-algebra on X. (b) For each set $E \cup Z \in \overline{\Sigma}$, define

$$\overline{\mu}(E \cup Z) = \mu(E).$$

Then $\overline{\mu}$ is a well-defined function on $\overline{\Sigma}$.

(c) $\overline{\mu}$ is a measure on $(X, \overline{\Sigma})$.

(d) $\overline{\mu}$ is the *unique* measure on $(X, \overline{\Sigma})$ that coincides with μ on Σ.

$\overline{\mu}$ is complete.

8.5 LET US SUMUP

In this unit we discussed the following

cantor lebesgue finctions

Lemma’s and theorems.

completeness of a measure spaces

8.6 KEYWORDS

Lemma- Lemma is minor, proven proposition which is used as a stepping stone to a larger result. For that reason, it is also known as a "helping theorem" or an "auxiliary theorem".

Theorem-A theorem is a statement that can be demonstrated to be true by accepted mathematical operations and arguments. In general, a theorem is an embodiment of some general principle that makes it part of a larger theory.

8.7 QUESTIONS FOR REVIEW

1) Let C be the Cantor set and ϕ the Cantor–Lebesgue function. Define $g(x) = \phi(x) + x$, and prove the following statements.

(a) Both g: $[0,1] \to [0,2]$ and g^{-1}: $[0,2] \to [0,1]$ are continuous, strictly increasing bijections.

(b) $g(C)$ is a closed subset of $[0,2]$, and $|g(C)| = 1$.

(c) Let N be a nonmeasurable subset of $g(C)$ (such a set exists by Problem 1.32). Then $A = g^{-1}(N)$ is Lebesgue measurable.

2) Each function ϕ_k is monotone increasing on the interval $[0,1]$, and $|\phi_{k+1}(x) - \phi_k(x)| < 2^{-k}$ for every $x \in [0,1]$.

3) The functions ϕ_k converge uniformly on $[0,1]$, and the limit function $\phi(x) = \lim_{k\to\infty} \phi_k(x)$ is continuous on $[0,1]$. Moreover, ϕ is differentiable at almost every point $x \in [0,1]$, and although ϕ is not differentiable at all points, we have $\phi'(x) = 0$ a.e. in $[0,1]$.

4) Let $\mathcal{B}_{\mathbb{R}^d}$ be the Borel σ-algebra on $\mathbb{R}^d$, and let μ be Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. Since every open subset of $\mathbb{R}^d$ is Lebesgue measurable, $\mathcal{B}_{\mathbb{R}^d}$ is contained in the σ-algebra $\mathcal{L}_{\mathbb{R}^d}$ of Lebesgue measurable subsets of $\mathbb{R}^d$. By Theorem 1.37, the σ-algebra $\overline{\mathcal{B}_{\mathbb{R}^d}}$ constructed in 1.1 is precisely $\mathcal{L}_{\mathbb{R}^d}$, and $\bar{\mu}$ is Lebesgue measure $|\cdot|$ on $(\mathbb{R}^d, \mathcal{L}_{\mathbb{R}^d})$. $\diamondsuit$

5) *Consider* the δ-measure as a measure on $(\mathbb{R}^d, \mathcal{B}_{\mathbb{R}^d})$. In this case $\overline{\mathcal{B}_{\mathbb{R}^d}} = \mathcal{P}(\mathbb{R}^d)$, and $\bar{\delta} = \delta$ on $(\mathbb{R}^d, \mathcal{P}(\mathbb{R}^d))$.

8.8 SUGGESTED READINGS AND REFERENCES

Fundamentals of Real Analysis, S K. Berberian, Springer.

An introduction to measure theory Terence Tao

Measure Theory Authors: **Bogachev**, Vladimir I

Chovanec Ferdinand. Cantor sets. Sci. Military J. 2010

Christopher Shaver. An exploration of the cantor set. Rose-Hulman Undergraduate Mathematics Journal.

Dauben Joseph Warren, Corinthians I. Georg cantor: The battle for transfinite set theory. American Mathematical Society.

Su Francis E, et al. Devil's staircase. Math Fun Facts.

8.9 ANSWERS TO CHECK YOUR PROGRESS

1 .Check section 8.3. For answer to check your progress 1.1

2 .Check section 9.3 for check your progress to 1.2

UNIT 9 LIMIT OF SEQUENCES OF SETS

STRUCTURE

9.1 OBJECTIVE

After going through this unit, you will be able to:

- Understandwhatsequence andseries offunctionis

Explainuniformconvergence

In this unit we discuss about limit of sequences ,limit superior and limit inferior.

9.2 INTRODUCTION

The study of advanced calculus is based on the thorough understanding of sequences and real numbers. There are various kinds of sequences such as bounded and monotonic sequences. A sequence (a_n) of real numbers is said to be bounded above if there exists a real number $M \in R$ such that $a_n \leq M$ for every $n \in N$. A sequence (a_n) is said to be bounded below if there exists a real number $m \in R$ such that $m \leq a_n$ for every $n \in N$. A sequence (a_n) is said to be bounded if it isboth bounded above and bounded below. A sequence (a_n) is monotonic increasingif

$a_{n+1} \geq a_n$ for all $n \in N$. The sequence is strictlymonotonicincreasing if we have

>in the definition. Monotonic decreasing sequences are defined similarly.

The limit of a sequence is the value that the terms of a sequence ”tend to”. If such a limit exists, the sequence is called convergent.Asequence which does not converge is said to be divergent. The limit of a sequence is said to be the fundamental notion on which the whole of analysis ultimately rests. Asequence is said to be convergent if it approaches some limit. A sequence convergeswhenit keepsgettingcloserandcloser to acertainvalue.Asequence $\{fn\}$ of functions is said to converge point wise on a set S to a limit function f, if for each $x \in S$ andfor each $\varepsilon > 0$ there exists an N (depending on x and ε) such that, ½$fn(x) - f(x)$½<
ε, for all $n > N$. Asequence of real valued functions $<fn>$ defined on a set S is said to converge uniformlyto a real valued function f on S if for $\varepsilon > 0$ $\exists m \in N$ such that
$|fn(x) - f(x)| < \varepsilon$ $n \geq m$ and $x \in S$

In this unit, you will studyabout sequences and series of function, uniform convergence indetail.

In mathematics, the **limit** of a sequence of sets A_1, A_2, ... (subsets of a common set X) is a set whose elements are determined by the sequence in either of two equivalent ways: **(1)** by upper and lower bounds on the sequence that converge monotonically to the same set (analogous to convergence of real-valued sequences) and **(2)** by convergence of a sequence of indicator functions which are themselves real-valued. As is the case with sequences of other objects, convergence is not necessary or even usual.

More generally, again analogous to real-valued sequences, the less restrictive **limit infimum** and **limit supremum** of a set sequence always exist and can be used to determine convergence: the limit exists if the limit infimum and limit supremum are identical. (See below). Such set limits are essential in measure theory and probability.

It is a common misconception that the limits infimum and supremum described here involve sets of accumulation points, that is, sets of $x = \lim_{k\to\infty} x_k$, where each x_k is in some A_{nk}. This is only true if convergence is determined by the discrete metric (that is, $x_n \to x$ if there is N such that $x_n = x$ for all $n \geq N$).

9.3 SEQUENCES AND SERIES OF FUNCTIONS

A sequence is a function whose domain is the set of natural numbers. If the codomain is the set □ of real numbers, it is called a real sequence; if it is theset □ of complexnumbers, it is called a complex sequence and likewise if it is a set of polynomials, it is a sequence of polynomials.

The image of the numbers 1, 2, 3, ... are called the first, second, third terms of thesequence, respectively.

Thusa real sequence can be conceived as a collection of numbers so that to every natural number there is a unique member of that collection. If the natural number is n, the corresponding number is denoted by x_n or y_n or z_n or u_n etc., and is called the nth term of the sequence. The sequence is denoted by$\{x_n\}$.

Thus $x_n = \frac{1}{n}$

is a sequence whose 1st, 2nd, 3rd terms are respectively 1, $\frac{1}{2}, \frac{1}{3}$. This sequence is called the *harmonic sequence*.

Another example of a sequence is $y_n = (-1)^n$. The first few terms of the sequence are {– 1, 1, – 1, 1, ...}.

The sequence $Z_n = 5$ is also a sequence, each of its term being 5. Such a sequence is called a *constant sequence*.

Bounded and Unbounded Sequences

A sequence $\{x_n\}$ is said to be *boundedabove* if all its terms are less than or equal to a real number, i.e., there exists $K \in$ □ such that $x_n \leq K$ for all $n \in$ □.

As for example, the sequence $\left\{\frac{1}{n}\right\}$ is bounded above since $\frac{1}{n} \leq 1$ for all $n \in$ □, the sequence $\left\{\frac{5n+1}{2n+2}\right\}$ is bounded above since $\frac{5n+1}{3n+2} \leq 3$ for all n, but the sequence $\{n^2\}$ is not bounded above since there exists no such real number K so that $n^2 \leq K$ for all n. In fact it is easy to observe that for every real number K there is an n such that $n^2 > K$. Such a sequence as above is called an unbounded sequence.

A sequence $\{x_n\}$ is said to be *bounded below* if all its terms are greater than or equal to a real number, i.e., there exists $K \in$ □ such that $x_n \geq k$ for all $n \in$ □. The sequence $\left\{\frac{1}{n}\right\}$ is bounded below since $\frac{1}{n} \geq 0$ for all n. The sequence $\left\{\frac{5n+1}{3n+2}\right\}$ is also bounded below since $\frac{5n+1}{3n+2} \geq 0$ for all n. The sequence $\{(-$

z1)^{n}5} is bounded below since $(-1)^n 5 \geq -5$ for all $n \in$, but the sequence $\{(-2)^n\}$is not bounded below since there is no such real number k for which $k \leq (-2)^n$. Indeed, if K is a negative real number, there always exists, an (odd) integer n such that $(-2)^n < k$.

A sequence is said to be *bounded* if it is bounded both above and below, i.e., if there exist $K, k \in$ such that $k \leq x_n \leq K$ for all $n \in N$.

The numbers K and k are called respectively an upper bound and a lower bound of the sequence $\{x_n\}$. Note that if a sequence $\{x_n\}$ has an upper bound, it has many upper bounds; similarly if a sequence $\{x_n\}$ has a lower bound, it has

many lower bounds. For example, for the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ just as 3 is an

upper bound, any real number greater than 3 is also an upper bound.

Monotone Sequence

A sequence $\{x_n\}$ is said to be monotone increasing if $x_n \leq x_{n+1}$ for every $n \in$; the sequence is called *strictly increasing* if $x_n < x_{n+1}$ for every $n \in$. Clearly the sequence $\{n^2\}$ is monotone (strictly) increasing since $n^2 \leq (n + 1)^2$ always.

The sequence $\{(-2)^n\}$ is not monotone increasing since $(-2)^2 \not\leq (-2)^3$.

A sequence $\{x_n\}$ is said to be *monotone decreasing* if $x_{n+1} \leq x_n$ for every n $\in$; the sequence is called *strictly decreasing* if $x_{n+1} < x_n$ for every $n \in$.

The sequence $\left\{\frac{1}{n^2+1}\right\}$ is monotone (strictly) decreasing as

1 1

$(n$ 1)^2

1 n^2

1

for every n. The sequence $\{-n^3\}$ is strictly decreasing as $\left(-\frac{1}{2}\right)^n$

$-(n+1)^3 < -n^3$ but the sequence

is not monotone or strictly decreasing

as $\left(-\frac{1}{2}\right)^4 \not< \left(-\frac{1}{2}\right)^3$.

Convergent Sequence

A very natural inquiry about a sequence $\{x_n\}$ is whether the terms x_n come close to any real number when n is very very large. This is what is known as the convergence of a sequence.

Definition: Asequence$\{x_n\}$ is said to *converge* to a real number l if for every $\varepsilon > 0$, there exists $n_0 \in$ such that

$$|x_n - l| < \varepsilon \quad \text{for every } n \geq n_0$$

The number l is called *limit* of the sequence $\{x_n\}$.

The fact that $\{x_n\}$ converges to l is expressed symbolically by $\lim_{n\to\infty} x_n = l$.

A sequence $\{x_n\}$ is called *convergent* if it converges to a limit l. A sequence which converges to zero is called a *null sequence.*

Thefollowingfactsfollowreadilyfromthe definition:

Fact 1 : A sequence may or may not converge.

Fact 2 : If a sequence is convergent, it converges to a unique limit, i.e., it cannotconvergetotwodifferentlimits.

Fact 3 : Everyconvergent sequence is always bounded, but not conversely.

Proof: Let $\{x_n\}$ be a convergent sequence with limit l. Then for a given ε $(> 0) = l$, say, there exists a positive integer n_0 such that

$| x_n - l | < l$ for all $n \geq n_0$

i.e., $l - 1 < x_n < l + 1$ for all $n \geq n_0$

Fact 4 : A monotone increasing sequence bounded above is always convergent and converges to its least upper bound.

Fact 5 : A monotone decreasing sequence bounded below is always convergent and converges to its greatest lower bound.

Fact 6 : Every constant sequence is convergent. Let $L = \min \{x_1, x_2, ..., x_{n_0}, | l | - 1\} \in$ □

and $U = \max \{x_1, x_2, ..., x_{n_0}, | l | + 1\} \in$ □

then $L < x_n < U$ for all n. Hence $\{x_n\}$ in a bounded sequence.

But the converse of this theorem is not true.

For example, the sequence $\{1 + (-1)^n\}$ is bounded but it does not converges to anyfinite limit. If the sequence is {0, 2, 0, 2, ...} then its lower bound is 0 and upper bound is 2.

Cauchy's Criterion of Convergence

Since proof of convergence of a sequence requires determination of the limit, proving convergence is not always easy. Cauchy therefore provided an alternative way to prove convergence of a sequence, called Cauchy's criterion which avoids the determination of the limit. This maybe stated as follows:

A sequence $\{x_n\}$ is convergent iff, for every $\varepsilon > 0$, there exists $n_0 \in$ □, usuallydependingon ε, such that

$| x_m - x_n | < \varepsilon$ for all $m, n \geq n_0$.

or equivalently, $| x_{n+p} - x_n | < \varepsilon$ for all $n \ge n_0$, $p = 0, 1, 2, 3, ...$

The sequence $\left\{\frac{1}{n}\right\}$ is convergent since

$$\left|\frac{1}{n+p} - \frac{1}{n}\right| < \varepsilon \text{ if } \frac{p}{n(n+p)} < \varepsilon$$

i.e., if $\frac{1}{n} < \varepsilon$, i.e., if $n > \frac{1}{\varepsilon}$, i.e., if $n \ge n_0 = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}$

(Observe that $n \ge \left[\frac{1}{\varepsilon}\right] + 1 \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow \frac{p}{n(n+p)} < \varepsilon \Rightarrow \left|\frac{1}{n+p} - \frac{1}{n}\right| < \varepsilon$)

Example 4.1: Show that the sequence $\{x_n\}$ is convergent when

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

Solution: Observe

$$\frac{1}{n!} = \frac{1}{1.2.3.\ldots.n} < \frac{1}{2^{n-1}}$$

For $m > n$

$$|x_m - x_n| = \left|\frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots + \frac{1}{m!}\right| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}}$$

$$= \frac{1}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right)$$

$$= \frac{1}{2^n} \cdot \frac{1 - \left(\frac{1}{2}\right)^{m-n}}{1 - \frac{1}{2}} < \frac{1}{2^n} \cdot 2 = \frac{1}{2^{n-1}} < \varepsilon$$

$\to 0$ as $n \to \infty$

Hence $\{x_n\}$ is convergent.

Algebra of Limits

The followng result is of immense importance in evaluation of limits.

Theorem 4.1: If $\lim_{n\to\infty} x_n = l$ and $\lim_{n\to\infty} y_n = m$, then

(i) $\lim_{n\to\infty}\{x_n + y_n\} = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n = l + m$.

(ii) $\lim_{n\to\infty}\{x_n - y_n\} = \lim_{n\to\infty} x_n - \lim_{n\to\infty} y_n = l - m$.

(iii) $\lim_{n\to\infty}\{x_n y_n\} = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n = l \cdot m$

(iv) $\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n} = \frac{l}{m}$

if $m \neq 0$, provided the above limits exist.

Anotherresultplaysa dominantrole in manysituations. This is the so called *sandwich theorem* stated as follows:

Theorem 4.2: (a) If $x_n < y_n$ for all $n \in$ □, then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.

(b) If $x_n < y_n < z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = l$, then $\lim_{n\to\infty} y_n = l$.

The proofs of the above theorems are outside the scope of this text.

Example 4.2: Show that the sequence $\left\{\frac{2n+3}{3n-2}\right\}$ is convergent.

Solution: Since $-4 < 9$, $6n - 4 < 6n + 9$. or $2(3n - 2) < 3(2n + 3)$

or $\frac{2n+3}{3n-2} > \frac{2}{3}$

Hence the sequence $\left\{\frac{2n+3}{3n-2}\right\}$ is bounded below.

Further taking, $x_n = \frac{2n+3}{3n-2}$, weobserve

$$x_n - x_{n+1} = \frac{2n+3}{3n-2} - \frac{2(n+1)+3}{3(n+1)-2}$$

$$= \frac{(2n+3)(3n-1)-(2n+5)(3n-2)}{}$$

=

i.e., $x_{n+1} \le x_n$ for all n.

$$= \left\{\frac{2n+3}{3n-2}\right\} \quad \frac{6n^2+11n+3-6n^2-11n+10}{(3n-2)(3n+1)} \quad \frac{13}{(3n-2)(3n+1)} \ge 0$$

for all n

Thus convergent.

being monotone decreasing and bounded below is

Divergent and Oscillatory Sequences

A sequence may be such that its terms become successively larger and larger, ultimately exceeding any big number. Such a sequence is said to diverge to

$+\infty$. On the other hand, a sequence may have decreasing terms so that ultimately it becomes smaller than any negative but numerically large real number. Such a sequence is said to diverge to $-\infty$. Such sequences are also possible the terms of which do not approach any definite real number nor do exceed any large positive real number or recede any arbitrary negative number. These are nothing but oscillatory sequences. The formal definitions go as follows:

Definition: A sequence $\{x_n\}$ is said to *diverge* to $+\infty$ if for every large $G > 0$, there exists $n_0 \in$ □ such that

$$x_n \ge G \text{ for all } n \ge n_0.$$

The fact $\{x_n\}$ diverges to ∞ is expressed symbolically by $\lim_{n\to\infty} x_n = \infty$.

A sequence $\{x_n\}$ is said to diverge to $-\infty$ if for every large $G > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$x_n \leq -G \quad \text{for all } n \geq n_0.$$

This is expressed symbolically by $\lim\limits_{n \to \infty} x_n = -\infty$.

A non-constant sequence which is bounded and not convergent is a finitely oscillatory sequence and a non-constant sequence which is unbounded and not convergent is an infinitely oscillatory sequence. For example, the sequence $x_n = 5 - (-1)^n 2$ is a finitely oscillatory sequence but the sequence $y_n = (-2)^n$ is an infinitely oscillatory sequence.

Theorem 4.3: If $\{x_n\}$ be a sequence such that $\lim\limits_{n \to \infty} \left| \dfrac{x_{n+1}}{x_n} \right| = l$ where $0 \leq l < 1$, then the sequence $\{x_n\}$ is a null sequence, i.e., $\lim\limits_{n \to \infty} x_n = 0$.

Proof: Beyond the scope of this book.

Example 4.3: Prove that $\lim\limits_{n \to \infty} \dfrac{x^n}{\lfloor n} = 0$ for every real value of x.

Solution: Here $x_n = \dfrac{x^n}{\lfloor n}$

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{x^{n+1}}{\lfloor n+1} \times \frac{\lfloor n}{x^n} \right| = \left| \frac{x}{n+1} \right|$$

and $x_{n+1} = \dfrac{(x)^{n+1}}{|n+1}$

$$= \frac{|x|}{n+1} \to 0$$

as $n \to \infty$ for all real value of x.

$$\therefore \quad \lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| = 0$$

Hence $\lim_{n\to\infty} \dfrac{x^n}{|n} = 0$.

Example 4.4: Prove that $\lim_{n\to\infty} \dfrac{x^n}{n} = 0$ if $|x| < 1$.

Solution: Here $x_n = \dfrac{x^n}{n}$

and $x_{n+1} = \dfrac{x^{n+1}}{n+1}$

$$\therefore \quad \left|\frac{x_{n+1}}{x_n}\right| = \left|\frac{x^{n+1}}{n+1} \times \frac{n}{x^n}\right| = \left|\frac{nx}{n+1}\right| = \frac{n}{n+1}|x|$$

$$\therefore \quad \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{1+\frac{1}{n}}|x| \to |x| \text{ as } n \to \infty$$

$$\lim_{n\to\infty} \frac{x^n}{n} = 0 \quad \text{if} \quad |x| < 1.$$

n

When $x = 1$, the given sequence is a harmonic sequence which converges to zer(o1)^n

as $n \to \infty$ and when $x = -1$, the givensequence is as $n \to \infty$.

$_n$ which converges to zero

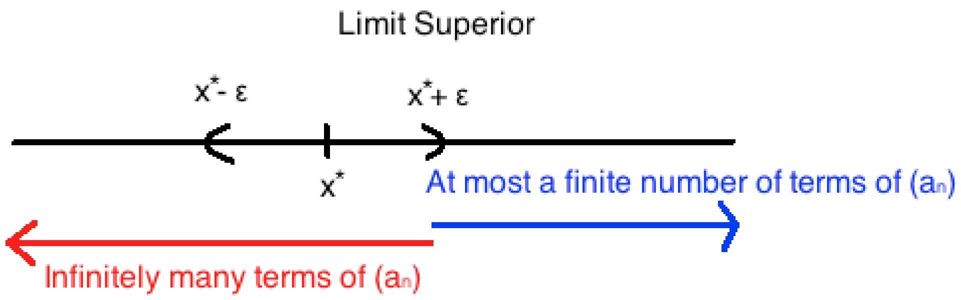
Hence $\lim_{n\to\infty} x^n$ =

$_n$ n

9.4 LIMIT SUPERIOR AND LIMIT INFERIOR

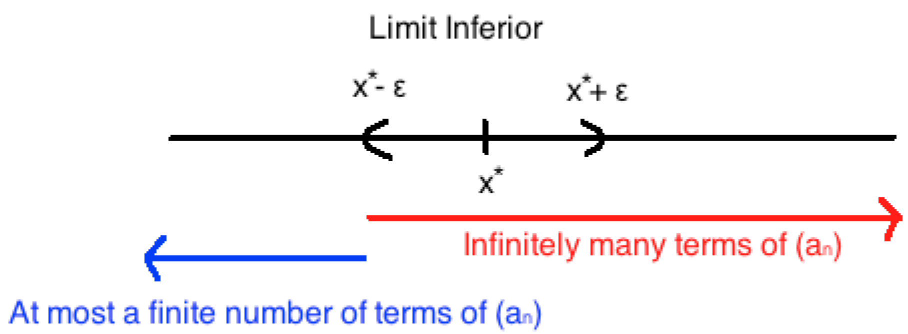
Definition: If (a_n) is a bounded sequence, then the **limit superior** of (a_n) is a real number x^* denoted $\limsup_{n\to\infty} a_n = x^*$ such that $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $a_n < x^* + \epsilon$ and there are infinitely many terms of a_n in $V_\epsilon(x^*)$. Similarly, the **limit inferior** is a real number x_* denoted $\liminf a_n = x_*$ such that $\forall \epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $x_* - \epsilon < a_n$ and there are infinitely many terms of a_n in $V_\epsilon(x_*)$.

Let's first look at the limit superior of a sequence:



From the definition of the limit superior of a bounded sequence, then for all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $a_n < x^* + \epsilon$.

Therefore, given some positive ϵ we can find a natural number N such that all successive terms a_n are less than $x^* + \epsilon$. Therefore, for finite first few terms up until N it is possible that $x^* + \epsilon < a_n$ but since there are only a finite number of terms for which this can happen, it follows that there are only a finite number of terms a_n such that $x^* + \epsilon < a_n$ for any $\epsilon > 0$.



The limit superior of a sequence is analogous. For all $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that if $n \geq N$ then $x * - \epsilon < a_n$. Therefore, given some positive ϵ we can find a natural number N such that all successive terms a_n are greater than $x * - \epsilon$. Therefore, for the finite first fewterms up until N it is possible that $a_n < x * - \epsilon$ but since there are only a finite number of terms for which this can happen, it follows that there are only a finite number of terms a_n such that $a_n < x^* - \epsilon$.

Theorem 1: Let (x_n) be a bounded sequence. The following statements are equivalent:

a) $x^* = \limsup x_n$.

b) If $A = \{a : a \text{ is an accumulation point of } (x_n).\}$ then $x^* = \sup A$.

c) If $B = \{x : x < x_n \text{ for at most finitely many } n \in \mathbb{N}.\}$ then $x^* = \inf B$.

Proof Let (x_n) be a bounded sequence of real numbers.

a) $\Longrightarrow$ b)

Let $x^* = \limsup x_n$, and let $s = \sup A$.

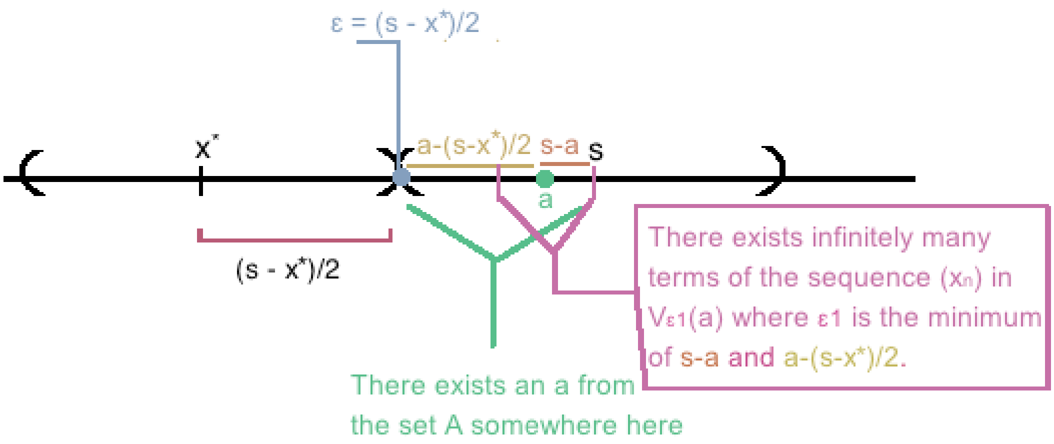
We ultimately want to show that $x^* = s$. First note that x^* is an accumulation point of the sequence (x_n) since $\forall \epsilon > 0$, by the definition of the limit superior, $V_\epsilon(x^*)$ contains infinitely many terms of (x_n) and eventually for some $N \in \mathbb{N}$ if $n \geq N$ then all successive terms x_n are contained in $V_\epsilon(x^*)$.

Therefore $x^* \in A$ and so $x^* \leq s = \sup A$ by the definition that s is the supremum of A.

- Now we will show that it is not possible that $x^* < s$ which will force $x^* = s$.

- Suppose that $x^* < s$. Then it follows that $s - x^* > 0$ and so $\frac{s - x^*}{2} > 0$. Let $\epsilon = \frac{s - x^*}{2}$.

- Now since $s = \sup A$ (as a reminder A is the set of accumulation points of the sequence (x_n)), then it follows that since $\frac{s - x^*}{2} < s$ that there exists an accumulation point $a \in A$ such that $\frac{s - x^*}{2} < a \leq s$. By the definition of an accumulation point of a sequence (x_n) there exists a subsequence of (x_n), call it (x_{n_k}) such that $\forall \epsilon > 0$ and for all $N \in \mathbb{N}$ there exists an $n \geq N$ such that x_n is in $V_\epsilon\, a$. Let $\epsilon_1 = \min\{a - \frac{s - x^*}{2}, s - a\}$. Then there exists infinitely many terms of (x_n) in $V_{\epsilon_1}(a)$.



- But this is a contradiction to the fact that $x* = \lim\sup x_n$ since then there exists infinitely many terms to the right of $x^*+\epsilon$, in other words, there does not exist an $N\in\mathbb{N}$ such that $\forall n\geq N$ then $x_n<x^*+\epsilon$. Thus it cannot be that $x^*<s$ and so $x^*=s=\sup A$.

- b)⟹c). Consider the set $B=\{x:x<x_n$ for at most finitely many $n\in\mathbb{N}\}$. Notice that for all $\epsilon>0$ we have that $(x^*+\epsilon)\in B$ since there are only a finite number of terms x_n such that $x^*+\epsilon<x_n$. Thus there are an infinite number of terms x_n such that $x^*-\epsilon<x_n<x^*$, and thus $\forall\epsilon>0$, $(x^*-\epsilon)\notin B$. Since $x^*=\sup A$ it follows that then $x^*=\sup A=\inf B$.

Theorem. Let {Ai}{Ai} be a sequence of sets with i∈Z+={1,2,…}i∈ℤ+={1,2,…}. Then

1. for II ranging over all infinite subsets of Z+ℤ+,

.

lim
supAi=⋃I⋂i∈ IAi,lim
sup Ai=⋃I⋂i∈ IAi,

.

2. for II ranging over all subsets of Z+ℤ+ with finite compliment,

lim
infAi=⋃I⋂i∈ IAi,lim
inf Ai=⋃I⋂i∈ IAi,

3. lim infAi⊆lim supAilim infAi⊆lim supAi.

Proof.

1. We need to show, for II ranging over all infinite subsets of Z+ℤ+,

⋃I⋂i∈ IAi=∞⋂n=1∞⋃i=nAk.⋃I⋂i∈ IAi=⋂n=1∞⋃i=n∞Ak. (1)

Let xx be an element of the LHS, the left hand side of Equation (1). Then x∈ ⋂i∈ IAix∈ ⋂i∈ IAi for some infinite subset I⊆Z+I⊆ℤ+. Certainly, x∈ ⋃∞i=1Aix∈ ⋃i=1∞Ai. Now, suppose x∈ ⋃∞i=kAix∈ ⋃i=k∞Ai. Since II is infinite, we can find an l∈ Il∈ I such that l>kl>k. Being a member of II, we have that x∈ Al⊆⋃∞i=k+1Aix∈ Al⊆⋃i=k+1∞Ai. By induction, we have x∈ ⋃∞i=nAix∈ ⋃i=n∞Ai for all n∈ Z+n∈ ℤ+. Thus xx is an element of the RHS. This proves one side of the inclusion (⊆⊆) in (1).

To show the other inclusion, let xx be an element of the RHS. So x∈ ⋃∞i=nAix∈ ⋃i=n∞Ai for all n∈ Z+n∈ ℤ+ In ⋃∞i=1Ai⋃i=1∞Ai, pick the least element n0n0 such that x∈ An0x∈ An0. Next, in ⋃∞i=n0+1Ai⋃i=n0+1∞Ai, pick the least n1n1 such that x∈ An1x∈ An1. Then the set I={n0,n1,…}I={n0,n1,…} fulfills the requirement x∈ ⋂i∈ IAix∈ ⋂i∈ IAi, showing the other inclusion (⊇⊇).

2. Here we have to show, for II ranging over all subsets of Z+ℤ+ with Z+−Iℤ+-I finite,

.

⋃I⋂i∈ IAi=∞⋃n=1∞⋂i=nAk.⋃I⋂i∈IAi=⋃n=1∞⋂i=n∞Ak. (2)

Suppose first that xx is an element of the LHS so that x∈ ⋂i∈ IAix∈ ⋂i∈ IAi for some II with Z+−IℤZ+-I finite. Let n0n0 be a upper bound of the finite set Z+−IℤZ+-I such that for any n∈ Z+−In∈ ℤZ+-I, n<n0n<n0. This means that any m≥n0m≥n0, we have m∈ Im∈ I. Therefore, x∈ ⋂∞i=n0Aix∈ ⋂i=n0∞Ai and xx is an element of the RHS.

Next, suppose xx is an element of the RHS so that x∈ ⋂∞k=nAkx∈ ⋂k=n∞Ak for some nn. Then the set I={n0,n0+1,…}I={n0,n0+1,…} is a subset of Z+ℤZ+ with finite complement that does the job for the LHS.

3. The set of all subsets (of Z+ℤZ+) with finite complement is a subset of the set of all infinite subsets. The third assertion is now clear from the previous two propositions. QED

.

Corollary. If {Ai}{Ai} is a decreasing sequence of sets, then

lim infAi=lim
supAi=limAi=⋂Ai.lim
inf Ai=lim
sup Ai=lim Ai=⋂Ai.

Similarly, if {Ai}{Ai} is an increasing sequence of sets, then

lim infAi=lim
supAi=limAi=⋃Ai.lim

inf⁡Ai=lim

sup⁡Ai=lim⁡Ai=⋃Ai.

Proof. We shall only show the case when we have a descending chain of sets, since the other case is completely analogous.

Let A1⊇A2⊇…A1⊇A2⊇… be a descending chain of sets.

Set A=⋂∞i=1AiA=⋂i=1∞Ai. We shall show that

lim supAi=lim

infAi=limAi=A.lim

sup⁡Ai=lim

inf⁡Ai=lim⁡Ai=A.

First, by the definition of of a sequence of sets:

lim

supAi=∞∩n=1∞∪i=nAk=∞∩n=1An=A.lim

sup⁡Ai=⋂n=1∞⋃i=n∞Ak=⋂n=1∞An=A.

Now, by Assertion 3 of the above Theorem, lim infAi⊆lim supAi=Alim

inf⁡Ai⊆lim sup⁡Ai=A, so we only need to show that A⊆lim

infAiA⊆lim inf⁡Ai. But this is immediate from the definition of AA, being the intersection of all AiAi with subscripts iitaking on all values of Z+ℤ+. Its complement is the empty set, clearly finite. Having shown both the existence and equality of the and of the AiAi's, we conclude that the limit of AiAi's exist as well and it is equal to AA.

Check your progress

1.Example Calculate lim sup an and lim inf an for an = (−1)n(n + 5)/n.

9.5 LET US SUM UP

In this unit we discussed the following

- Limit of sequence of sets
- Limit superior and limit inferior

9.6 KEYWORDS

limit superior The **limit superior** of is the smallest real number such that, for any positive real number , there exists a natural number such that for all . In other words, any number larger than the **limit superior** is an eventual upper bound for the sequence.

Limit inferior The **limit inferior** of is the largest real number such that, for any positive real number , there exists a natural number such that for all . In other words, any number below the **limit inferior** is an eventual lower bound for the sequence

9.7 QUESTIONS FOR REVIEW

1.Example Calculate lim sup an and lim inf an for an = (−1)nn/(n + 8).

9.8 SUGGESTED READINGS AND REFERENCES

Fundamentals of Real Analysis, S K. Berberian, Springer.

An introduction to measure theory Terence Tao

Measure Theory Authors: **Bogachev**, Vladimir I

Chovanec Ferdinand. Cantor sets. Sci. Military J. 2010

Christopher Shaver. An exploration of the cantor set. Rose-Hulman Undergraduate Mathematics Journal.

Dauben Joseph Warren, Corinthians I. Georg cantor: The battle for transfinite set theory. American Mathematical Society.

Su Francis E, et al. Devil's staircase. Math Fun Facts. http://www.math.hmc.edu/funfacts, http://www.math.hmc.edu/funfacts

Amir D. Aczel, A Strange Wilderness the Lives of the Great Mathematicians, Sterling Publishing Co. 2011.

Planetmath.org

Proofwiki.Org

9.9 ANSWERS TO CHECK YOUR PROGRESS

1. Solution

Define $\alpha_n = \sup \{a_k \mid k \geq n\}$. Then

$\alpha_n = \sup \{(-1)^n(n + 5)/n, (-1)^{n+1}(n + 6)/(n + 1), \ldots\} = (n+5)/n$ for n even, and $(n+6)/(n+1)$ for n odd

$\to 1$ as $n \to \infty$.

Therefore $\limsup a_n = 1$. Similarly $\liminf a_n = -1$.

UNIT 10 MEASURABLE FUNCTIONS THEOREMS

STRUCTURE

10.1 OBJECTIVES

Aftergoingthroughthisunit, you will be ableto:

- Understand measurablefunctions
- ExplainEgoroff's theorem
- DiscussLusin's theorem

10.2 INTRODUCTION

Measurable functions are functions that we can integrate with respect to measures in muchthesamewaythatcontinuousfunctionscan beintegrated"dx". Recall that the Riemann integral of a continuous function f over a bounded interval is defined as a limit of sums of lengths of subintervals times values of f on thesubintervals. Themeasureofa setgeneralizesthelengthwhileelements of the σ-field generalize the intervals. Recall that a real-valued function is continuous if and only if the inverse image of every open set is open. This generalizes to the inverse image of everymeasurableset beingmeasurable.

In other words we can say that , a measurable function is a function between two measurablespaces such that the preimage of any measurable set is measurable, analogously to the definition that a function between topological spaces is continuous if the preimage of each open set is open. In real analysis, measurable functions are used in the definition of the Lebesgue integral. In probability theory, a measurable function on a probabilityspace is known as a randomvariable.

In this unit, you will studyabout measurable functions, Egoroff's theorem and Lusin'stheorem in detail

10.3 MEASURABLE FUNCTIONS

Suppose X be a set and U be a σ-algebra on X.

Definition: The pair (X, U) is called a measurable space.

Definition: Let f be a function defined on a measurable space (X, U), with values in the extended real number system. The function f is called measurable if the set $\{x: f(x) > a\}$ is measurable for every real a.

Theorem 10.1: The conditions given below are equivalent:

1. $\{x: f(x) > a\}$ is measurable for every real a.
2. $\{x: f(x) \geq a\}$ is measurable for every real a.
3. $\{x: f(x) < a\}$ is measurable for everyreal a.
4. $\{x: f(x) \leq a\}$ is measurable for everyreal a.

Proof: The statement follows the equalities,

1. $\{x : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x : f(x) > a - \frac{1}{n}\}$

2. $\{x : f(x) < a\} = X \setminus \{x : f(x) \geq a\}$

3. $\{x : f(x) \le a\} = \bigcap_{n=1}^{\infty} \{x : f(x) < a + \frac{1}{n}\}$

4. $\{x : f(x) > a\} = X \setminus \{x : f(x) \le a\}$

Theorem 10.2: Let $f(n)$ be a sequence of measurable functions. For $x \in X$, put

$$g(x) = \sup_n f_n(x), \quad (n \in N)$$

$$h(x) = \limsup_{n \to \infty} f_n(x)$$

Then g and h are measurable.

Proof: Here, $\{x : g(x) \le a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) \le a\}$

Now as the Left hand side is measurable, it follows that the Right hand side is measurable too. The same proof works for inf. Now,

$h(x) = \inf g_m(x)$, where $g_m(x) = \sup_{n \ge m} f_n(x)$

Theorem 10.3: Let f and g be measurable real valued functions defined on X. Let F be a real and continuous function on R^2. Set $h(x) = F(f(x), g(x))$ $(x \in X)$. Then, h is measurable.

Proof: Suppose $G_a = \{(u, v): F(u, v) > a\}$. Then G_a is open subset of R^2, and hence $G_a = \bigcup_{n=1}^{\infty} I_n$, where (I_n) is a sequence of open intervals, $I_n = \{(u,v): a_n < u < b_n, c_n < v < d_n\}$.

The sets $\{x: a_n < f(x) < b_n\}$ and $\{x: (f(x), g(x)) \in I_n\} = \{x: a_n < f(x) < b_n\} \cap \{x: c_n < g(x) < d_n\}$ are measurable.

Hence, the same holds for

$$\{x: h(x) > a\} = \{x: (f(x), g(x)) \in G_a\} = \bigcup_{n=1}^{\infty} \{x: (f(x), g(x)) \in I_n\}$$

Corollary: Let f and g be measurable. Then, the followingfunctions are measurable:

1. $f + g$
2. $f.g$
3. $|f|$
4. f/g (if $g \neq 0$)
5. $\max\{f, g\}, \min\{f, g\}$

since, $\max\{f, g\} = 1/2(f + g + |f - g|)$ and $\min\{f, g\} = 1/2(f + g - |f - g|)$.

Definition: Let E be a measurable set and f be a function defined on E. Then f is said to be measurable (Lebesgue function) if for anyreal α, any one of the following fourconditionsis satisfied:

1. $\{x \mid f(x) > \alpha\}$ is measurable.
2. $\{x \mid f(x) \geq \alpha\}$ is measurable.
3. $\{x \mid f(x) < \alpha\}$ is measurable.
4. $\{x \mid f(x) \leq \alpha\}$ is measurable.

We will first prove that the above four conditions are equivalent.

(1) $\Leftrightarrow$ (4): Since,

$\{x \mid f(x) > \alpha\} = \{x \mid f(x) \leq \alpha\}^c$

and also we know that complement of a measurable set is measurable, therefore (1)$\Rightarrow$(4) and conversely.

(2) $\Leftrightarrow$ (3): Similarlysince (2) and (3) are complement of each other, (3) is measurable if (2) is measurableand conversely.

(1) $\Leftrightarrow$ (2): Now, it is sufficient to prove that (1)$\Rightarrow$(2) and conversely. Firstly, we show that (2)$\Rightarrow$(1).

The set $\{x \mid f(x) \geq \alpha\}$ is given to be measurable. Now,

$$\{x \mid f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \mid f(x) \geq \alpha + 1/n\}$$

But by(2), $\{x \mid f(x) \geq \alpha + 1/n\}$ is measurableand we know that countable union of measurablesets is measurable. Hence, $\{x \mid f(x) > \alpha\}$ is measurablewhich implies that (2)$\Rightarrow$(1). Conversely, let (1) holds. We have,

$$\{x \mid f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) > \alpha - 1/n\}$$

The set $\{x \mid f(x) > \alpha - 1/n\}$ is measurable by(1). Moreover, intersection of measurable sets is also measurable. Hence, $\{x \mid f(x) \geq \alpha\}$ is also measurable. Thus (1)$\Rightarrow$(2).

Hence, the four conditions are equivalent.

Lemma: If α is an extendedrealnumberthen, the abovefourconditionsimply that $\{x \mid f(x) = \alpha\}$ is also measurable.

Proof: Let α be a real number, then $\{x \mid f(x) = \alpha\} = \{x \mid f(x) \geq \alpha\} \cap \{x \mid f(x) \leq \alpha\}$.

Since $\{x \mid f(x) \geq \alpha\}$ and $\{x \mid f(x) \leq \alpha\}$ are measurable by conditions (2) and (4), the set $\{x \mid f(x) = \alpha\}$ is measurable being the intersection of measurable sets.

Let, $\alpha = +\infty$. Then,

$$\{x \mid f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) \geq n\}$$

which is measurablebythecondition(2)andbecausetheintersectionof measurable sets ismeasurable.

Similarlywhen $\alpha = -\infty$, then $\{x \mid f(x) = -\infty\} =$

again measurable by condition (4).

Hence proved.

10.3.1 Properties of Measurable Functions

$\bigcup_{n=1}^{\infty} \{x \mid f(x) \le -n\}$, which is

The set $\{x \mid f(x) > \alpha\}$ is inverse image of $(\alpha, \infty]$, where α is real. In the same way, the sets $\{x \mid f(x) \ge \alpha\}$, $\{x \mid f(x) < \alpha\}$ and $\{x \mid f(x) \in \alpha\}$ are inverse images of $[\alpha,$

$\infty]$, $[-\infty, \alpha)$ and $[-\infty, \alpha]$ respectively. Hence, we can also define a measurable function asfollows.

A function f defined on a measurable set E is said to be measurable if for anyreal α anyone of the four conditions is satisfied:

1. The inverse image $f^{-1}(\alpha, \infty]$ of the half-open interval $(\alpha, \infty]$ is measurable.
2. For every real α, the inverse image $f^{-1}[\alpha, \infty]$ of the closed interval $[\alpha, \infty]$ is measurable.
3. Theinverseimage $f^{-1}[-\infty, \alpha)$ of thehalfopeninterval $[-\infty, \alpha)$ is measurable.
4. The inverse image $f^{-1}[-\infty, \alpha]$ of the closed interval $[-\infty, \alpha]$ is measurable.

Notes:

1. A necessary and sufficient condition for measurability is that, $\{x \mid a \le f(x) \le b\}$ should be measurable for all a, b [including the case $a = -\infty$, $b = +\infty$], as anyset of this form can be written as the intersection of two sets, $\{x \mid f(x) \ge a\} \cap \{x \mid f(x) \le b\}$.

If f is measurable, each of these is measurable and so is $\{x \mid a \le f(x) \le b\}$. Conversely any set of the form occurring in the definition can easily be expressed in terms of the sets of the form $\{x \mid a \le f(x) \le b\}$.

2. As (α, ∞) is an open set, we can define a measurable function as a function f defined on a measurable set E, for which for every open set G in the real number system, $f^{-1}(G)$ is a measurable set.

Definition: Characteristic function of a set E is defined by,

$\chi(x) = \begin{cases} 1 \\ \end{cases}$

E $\lfloor 0$

if $x \in E$

if $x \notin E$

This is also known as indicator function.

Theorem 10.4: For any real c and two measurable real valued functions f, g the four functions $f + c$, cf, $f + g$ and fg are measurable.

Proof: We have that f is a measurable function and c is any real number. Then for any real number α,

$\{x \mid f(x) + c > \alpha\} = \{x \mid f(x) > \alpha - c\}$.

But, $\{x \mid f(x) > \alpha - c\}$ is measurable by the condition (1) of the definition. Hence $\{x \mid f(x) + c > \alpha\}$ and thus $f(x) + c$ is measurable.

Now, consider the function cf. When $c = 0$, cf is the constant function 0 and hence is measurable since, every constant function is continuous and so measurable. When $c > 0$, we have $\{x \mid cf(x) > \alpha\} = \{x \mid f(x) > \alpha/c\} = f^{-1}(\alpha/c, \infty]$, and so measurable. When $c < 0$, we have $\{x \mid cf(x) > r\} = \{x \mid f(x) < r/c\}$, and so measurable.

Now, if f and g are two measurable real valued functions defined on the same domain, we will show that $f + g$ is measurable. For this, it is sufficient to show that the set $\{x \mid f(x) + g(x) > \alpha\}$ is measurable.

If $f(x) + g(x) > \alpha$, then $f(x) > \alpha - g(x)$ and there is a rational number r such that,

$\alpha - g(x) < r < f(x)$

Since the functions f and g are measurable, the sets $\{x \mid f(x) > r\}$ and $\{x \mid g(x) > \alpha - r\}$ are measurable. Hence, their intersection, $S_r = \{x \mid f(x) > r\}$ $\cap$

$\{x \mid g(x) > \alpha - r\}$ is also measurable.

It can be shown that, $\{x \mid f(x) + g(x) > \alpha\} = U\{S_r$

$\mid r$ is rational$\}$.

As the set of rationals, is countable and countable union of measurable sets is measurable, therefore the set $U\{S_r \mid r$ is rational$\}$ and hence, $\{x \mid$

$f(x)$

$+ g(x) > \alpha\}$ is measurable which establishes that $f(x) + g(x)$ is measurable.

From this part it follows that $f - g = f + (-g)$ is also measurable, since when g is measurable $(-g)$ is also measurable. Next, we consider fg. The measurability of fg follows from the identity,

$$fg = \frac{1}{2}\left[(f + g)^2 - f^2 - g^2\right],$$

if we prove that f^2 is measurable when f is measurable. So, it is sufficient to prove that, $\{x \in E \mid f^2(x) > \alpha\}$, where α is a real number, is measurable.

Let, α be a negative real number. Then, the set $\{x \mid f^2(x) > \alpha\} = E$ (domain of the measurable function f). But, E is measurable by the definition of f. Hence $\{x \mid f^2(x) > \alpha\}$ is measurable when $\alpha < 0$.

Now let $\alpha \geq 0$, then $\{x \mid f^2(x) > \alpha\} = \{x \mid f(x) > \sqrt{\alpha}\} \cup \{x \mid f(x) < -\sqrt{\alpha}\}$.

Since f is measurable, it follows from this equality that $\{x \mid f^2(x) > \alpha\}$ is measurable for $\alpha \geq 0$. Hence, f^2 is also measurable when f is measurable. Therefore, the theorem follows from the above identity, since measurability of f and g imply the measurability of $f + g$.

From this we can also conclude that f/g $(g \neq 0)$ is also measurable.

Theorem 10.5: If f is measurable, then $|f|$ is also measurable.

Proof: It is sufficient to prove the measurability of the set,

$\{x : |f(x)| > \alpha\}$, where α is any real number.

If $\alpha < 0$, then $\{x : |f(x)| > \alpha\} = E$ (domain of f)

But E is assumed to the measurable. Hence $\{x : |f(x)| > \alpha\} = \{x : |f(x)| > \alpha\} \cup \{x \mid f(x) < -\alpha\}$

The right hand side of the equality is measurable since f is measurable.

Hence, $\{x : |f(x)| > \alpha\}$ is also measurable.

This proves the theorem.

Theorem 10.6: Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions. Then, $\sup\{f_1, f_2, \ldots, f_n\}$, $\inf\{f_1, f_2, \ldots, f_n\}$, $\sup_n f_n$, $\inf_n f_n$ and $\lim f_n$ are measurable.

Proof: Define a function $\phi(x) = \sup\{f_1, f_2, \ldots, f_n\}$.

We will prove that $\{x \mid \phi(x) > \alpha\}$ is measurable. In fact,

$$\{x \mid \phi(x) > \alpha\} = \bigcup_{i=1}^{n} \{x \mid f_i(x) > \alpha\}$$

Since each f_i is measurable, each of the set $\{x \mid f_i(x) > \alpha\}$ is measurable and therefore their union is also measurable. Hence, $\{x \mid \phi(x) > \alpha\}$ and so $\phi(x)$ is measurable. In the same way, define the function $m(x) = \inf\{f_1, f_2, \ldots, f_n\}$. Now since, $m(x) < \alpha$ iff $f_i(x) < \alpha$ for some i we have $\{x \mid m(x) < \alpha\} = \bigcup_{i=1}^{n} \{x \mid f_i(x) < \alpha\}$ and since $\{x \mid f_i(x) < \alpha\}$ is measurable on account of the measurability of f_i, we conclude that $\{x \mid m(x) < \alpha\}$ and so $m(x)$ is measurable.

Define a function,

$M'_n(x) = \sup_n f_n(x) = \sup\{f_1, f_2, \ldots, f_n\}$

We will now prove that the set,

$\{x \mid M'(x) > \alpha\}$ is measurable for any real α.

Now,

$\{x \mid M'(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > \alpha\}$ is measurable, since each f_n is measurable.

Similarly, if we define $m'(x) = \inf_n f_n(x)$, then

$\{x \mid m'(x) < \alpha\} = \bigcup_{i=1}^{\infty} \{x \mid f_n(x) < \alpha\}$

Therefore, measurabilityof f_n implies measurability of $m'(x)$. Now as

$\overline{\lim}\, f_n = \lim \sup f_n = \inf_n \left\{\sup_{n \geq k} f_n\right\}$

and

$\underline{\lim}\, f_n = \sup_n \left\{\inf_{n \geq k} f_n\right\}$

n k $\left(\right.$

$n \geq k \left.\right)$

the upper and lower limits are measurable.

Lastly, if the sequence is convergent, its limit is the common value of $\overline{\lim}\, f_n$ and $\underline{\lim}\, f_n$ and hence is measurable.

Definition: Let f and g be measurable functions. Then we define,

$f^+ = \text{Max}(f, 0)$

$f^- = \text{Max}(-f, 0)$

$f \vee g =$

$f + g + | f - g |$ 2

, i.e., Max(f, g)

and $f \wedge g =$

$\frac{f + g - | f - g |}{2}$, i.e., min($f$, g)

Theorem 10.7: Suppose f be a measurable function. Then, f and f are both measurable functions.

Proof: Let us suppose that $f > 0$. Then we have,

$f = f$ and $f = 0^+$ ...(10.1)

So in this case we have,

$f = {}^+ - f$ f –

Now, let us take f to be negative. Then,

$^+$

$f = \text{Max}(f, 0) = 0$

$f = \text{Max}(-f, 0) = -f$ ...(10.2)

Therefore on subtraction,

$f = f^+ - f^-$

In case $f = 0$, then

$f^+ = 0, \quad f^- = 0$ ...(10.3)

Therefore,

$f = f^+ - f^-$

Thus, for all f we have

$f = f^+ - f^-$ ...(10.4)

Also, adding the components of Equation (10.1) we have,

$f^+ + f^- = |f| = f^+ - f^-$ ...(10.5)

since, f is positive.

And from Equation (10.2) when f is negative we have,

$f^+ + f^- = 0 - f = -f = |f|$ ...(10.6)

In case f is zero, then

$f^+ + f^- = 0 + 0 = 0 = |f|$ ...(10.7)

That is for all f, we have

$|f| = f^+ - f^-$ ...(10.8)

Adding Equations (10.4) and (10.8) we have,

$f + |f| = 2f^+$

$\Rightarrow f^+ = 1/2\,(|f| - f)$ ...(10.9)

Similarly on subtracting, we obtain

$f^{-} = 1/2\,(|f| - f)$ ...(10.10)

Since, measurabilityof f implies themeasurabilityof $|f|$, it is obvious from Equations (10.9) and (10.10) that f^{+} and f^{-} are measurable.

Theorem 10.8: If f and g are two measurable functions, then $f \vee g$ and $f \wedge g$ are measurable.

Proof: We know that,

$$f \vee g = \frac{f + g + |f - g|}{2}$$

$$f \wedge g = \frac{f + g - |f - g|}{2}$$

Now, measurability of $f \Rightarrow$ measurability of $|f|$. Also if f and g are measurable, then measurable.

$f + g$,

$f - g$ are measurable. Hence, $f \vee g$ and $f \wedge g$ are

Definition: Astatement is said to hold almost everywhere in E, if and onlyif it holds everywhere in E except possiblyat a subset D of measure zero.

Examples:

1. Twofunctions f and g definedon E are said to be equalalmost everywhere in E, iff $f(x) = g(x)$ everywhere except a subset D of E of measure zero.
2. A function defined on E is said to be continuous almost everywhere in E, if and only if there exists a subset D of E of measure zero such that, f is continuous at everypoint of $E - D$.

Theorem 10.9: (a) If f is a measurable function on the set E and $E_1 \subset E$ is measurable set, then f is a measurable function on E_1.

(b) If f is a measurable function on each of the sets in a countable collection $\{E_i\}$ of disjoint measurable sets, then f is measurable.

Proof:

(a) For any real α, we have $\{x \in E_1; f(x) > \alpha\} = \{x \in E; f(x) > \alpha\} \cap E_1$. The result follows as the set on the right hand side is measurable.

(*b*) Write $E = \bigcup_{i=1}^{\infty} E_i$. Clearly, E, being the union of measurable set is measurable.

The result now follows, since for each real α, we have

$$E = \{x \in E;\ f(x) > \alpha\} = \bigcup_{i=1}^{\infty} E_i\, f(x) > \alpha\}.$$

Theorem 10.10: Suppose f and g be any two functions, which are equal almost everywhere in E. If f is measurable then g is also measurable.

Proof: Since f is measurable, for any real α the set $\{x \mid f(x) > \alpha\}$ is measurable. Now, we have to show that the set $\{x \mid g(x) > \alpha\}$ is measurable. For this, put

$E_1 = \{x \mid f(x) > \alpha\}$

$E_2 = \{x \mid g(x) > \alpha\}$

Consider the sets $E_1 - E_2$ and $E_2 - E_1$. Since $f = g$ almost everywhere, therefore measures of these sets are zero. That is, both of these sets are measurable. Now,

$E_2 = [E_1 \cup (E_2 - E_1)] - (E_1 - E_2)$

$= [E_1 \cup (E_2 - E_1)] \cap (E_1 - E_2)^c$

Since E_1, $E_2 - E_1$ and $(E_1 - E_2)^c$ are measurable, therefore we get that E_2 is measurable. Hence, the theorem is proved.

Corollary: Let, $\{f_n\}$ be a sequence of measurable functions such that $\lim_{n\to\infty} f_n = f$ almost everywhere. Then, f is a measurable function.

Proof: We have already proved that if $\{f_n\}$ is a sequence of measurable functions then $\lim_{n\to\infty} f_n$ is measurable. Also, it is given that $\lim_{n\to\infty} f_n = f$ almost everywhere.

Therefore, using the above theorem it follows that f is measurable.

Theorem 10.11: Characteristic function χ_A is measurable if and only if A is measurable.

Proof: Let A be measurable. Then,

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \text{ i.e., } x \in A^c \end{cases}$$

Hence, it is clear from the definition that domain of χ_A is $A \cup A^c$ which ismeasurable due to the measurability of A. Therefore, we need to prove that the set $\{x \mid \chi_A(x) > \alpha\}$ is measurable for any real α.

Let $\alpha \geq 0$. Then, $\{x \mid \chi_A(x) > \alpha\} = \{x \mid \chi_A(x) = 1\}$

$= A$(By the definition of characteristic function)

But, A is given to be measurable. Hence for $\alpha \geq 0$, the set $\{x \mid \chi_A(x) > \alpha\}$ is measurable. Now, let us take $\alpha < 0$. Then,

$\{x \mid \chi_A(x) > \alpha\} = A \cup A^c$

So $\{x \mid \chi_A(x) > \alpha\}$ is measurable for $\alpha < 0$ also, since $A \cup A^c$ has been proved to be measurable. Therefore if A is measurable, then χ_A is also measurable.

Conversely, let us suppose that $\chi_A(x)$ is measurable or the set $\{x \mid \chi_A(x) > \alpha\}$ is measurable for any real α. Let $\alpha \geq 0$. Then,

$\{x \mid \chi_A(x) > \alpha\} = \{x \mid \chi_A(x) = 1\} = A$

Therefore, measurabilityof $\{x \mid \chi_A(x) > \alpha\}$ implies that of the set A for $\alpha \geq 0$. Now consider $\alpha < 0$.Then,

$\{x \mid \chi_A(x) > \alpha\} = A \cup A^c$

Thus measurabilityof $\chi_A(x)$ implies measurabilityof the set $A \cup A^c$ which implies that the set A is measurable.

Theorem 10.12: If a function f is continuous almost everywhere in E, then it is measurable.

Proof: As f is continuous almost everywhere in E, therefore there exists a subset D of E with $m^*D = 0$ such that f is continuous at every point of the set $C = E - D$. To prove that f is measurable, let α denote any given real number.

It is sufficient to prove that the inverse image $B = f^{-1}(\alpha, \infty) = \{x \in E \mid f(x) > \alpha\}$ of the interval (α, ∞) is measurable. For doing this, let X denote an arbitrary point in $B \cap C$. Then, $f(x) > \alpha$ and f is continuous at X. Hence, there exists an open

x

interval U_x

Let,

containing X such that $f(y) > \alpha$ holds true for every point y of $E \cap U$.

$$U = \bigcup_{x \in B \cap C} U_x$$

Since $x \in E \cap U_x \subset B$ holds for every $x \in B \cap C$, we have

$B \cap C \subset E \cap U \subset B$

This implies,

$B = (E \cap U) \cup (B \cap D)$

As an open subset of R, U is measurable. Hence, $E \cup U$ is measurable. On the other hand, since $m^*(B \cap D) \leq m^*D = 0$, $B \cap D$ is also measurable. This implies that B is measurable. This completes the proof of the theorem.

Definition: A function ϕ, defined on a measurable set E, is called simple if there is a finite disjoint class $\{E_1, E_2, \ldots, E_n\}$ of measurable sets and a finite set $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of real numbers such that,

$$f(x) = \begin{cases} \alpha_i & \text{if } x \in E_i,\ i = 1, 2, \ldots, n \\ 0 & \text{if } x \notin E_1 \cup E_2 \cup \ldots \cup E_n \end{cases}$$

Thus, a function is simple if it is measurable and takes only a finite number of different values.

The simplest example of a simple function is the characteristic function χ_E of a measurable set E.

Definition: A function is said to be a step function if, $f(x) = C_i$, $\xi_{i-1} < x < \xi_i$ for some subdivision of $[a, b]$ and some constants C_i. Clearly, a step function is a simple function.

Theorem 10.13: Every simple function ϕ on E is a linear combination of characteristic functions of measurable subsets of E.

Proof: Let ϕ be a simple function and $c_1, c_2, \ldots, c_n$ denote the non zero real numbers in its image $\phi(E)$. For each $i = 1, 2, ..., n$

Let,

$$A_i = \{ x \in E : \phi(x) = C_i \}$$

Then we have,

$$\phi = \sum_{i=1}^{n} C_i \chi_{A_i}$$

On the other hand, if $\phi(E)$ contains no non zero real number, then $\phi = 0$ and is the characteristic function χ_φ of the empty subset of E.

10.3.2 Approximation of Measurable Functions by Sequence of Simple Functions

Definition: A function $s : X \to Y$ is a simple function if the range of s is a finite set. If s is a simple function and if $\{a_1, \ldots, a_n\}$ is the range of s, then we set $E_i =$

$s^{-1}(\{a_i\})$, $i = 1, 2, ..., n$. Thus,

$$s(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$$

This is known as the canonical representation of the simple function s.

A nonnegativesimplefunction is a simplefunction in whichtherangeis containedin

$[0,\infty)$. In particular, simple functions only take on finite values.

Notes:

1. s is measurable if and only if each E_k in the canonical representation is measurable.

2 . If $X = A\cup B$ and $A\cap B = \phi$, then $s = 1 = \chi_A + \chi_B$ is measurable irrespective of whether or not A and B are measurable. So if A, $B \neq \phi$, then A, $B \neq s^{-1}(\{y\})$ for any y in the range of s. Therefore, in the canonical representation we have $s(x)=\sum a_i\chi E_i(x)$, wherethe a_i aremutuallydistinct and the E_i are mutuallydisjoint.

Theorem 10.14: If (X, M) is a measurablespace and $f : X \to [0, \infty]$ is measurable, then there is a sequence $\{s_n\}$ of simple measurable functions such that,

(a) For each $x \in X$, $0 \leq s_1(x) \leq s_2(x) \leq ... \leq f(x)$.

(b) For each $x \in X$, $s_n(x) \to f(x)$ as $n \to \infty$.

(c) If $A \subseteq X$ is such that $f|A$ is bounded, then $s_n|A \to f|A$ uniformly.

Proof: Set $n \in N$. For $0 \leq k < n2^n$ we fix,

$F_{n,k} = [k2^{-n}, (k + 1)2^{-n})$. Then, $F_{n,k} \cap F_{n,j} = \phi$ for $k \neq j$ and

$$\bigcup_{k=0}^{n2^n-1} F_{n,k} = [0, n).$$

Define $\phi_n : [0, \infty] \to [0, \infty)$ by

$$\varphi_n(t) = \begin{cases} k2^{-n}, & \text{if } t \in F_{n,k} \text{ where } 0 \le k < n2^n \\ n, & \text{if } n \le t \le \infty \end{cases}$$

Then, each ϕ_n is Borel measurable

$$\varphi_n(x) = \sum_{k=0}^{n2^n-1} k\,2^{-n} \chi F_{n,k}(x) + n\chi[n, \infty](x)$$

Claim: For each n we have,

$\varphi_n(t) \le \varphi_{n+1}(t)$

Note that,

$\forall t \in [0, \infty]$.

$$F_{n,k} = [k2^{-n}, (k+1)2^{-n})$$
$$= [2k\,2^{-(n+1)}, 2(k+1)2^{-(n+1)})$$
$$= [2k\,2^{-(n+1)}, 2(k+1)2^{-(n+1)}) \cup [(2k+1)2^{-(n+1)}, (2k+2)2^{-(n+1)})$$

$$= \sup_{x \in A} | s_n(x) - f(x) | \le \sup_{t \in [0,M]} | \varphi_n(t) - t |$$

$$= F_{n+1,2k} \cup F_{n+1,2k+1}$$

Thus, if $0 \le t < n$, then $\exists k$ such that $t \in F_{n,}$.

Case 1: If $t \in F_{n+1,2k}$, then, $\varphi_n(t) = k2^{-n}$ and $\varphi_{n+1}(t) = 2k\,2^{-(n+1)} = \varphi_{n+1}(t) = 2k2^{-(n+1)} = \varphi_n(t)$.

Case 2: If $t \in F$, then, $\varphi(t) = k2^{-n}$ and φ

$\varphi_{n+1}(t) = \varphi_{n+1,2k+1}(t) = (2k+1)2^{-(n+1)} = \varphi_n(t)$

$+2^{-(n+1)} \geq \varphi_n(t)$.

If $t \geq n$, then $\phi_n(t) = n$ and $\phi_{n+1}(t) \geq n$. This proves the claim.

Also, note that for each $t \in [0, \infty]$ we have $\phi_n(t) \leq t$ and $\phi_n(t) \to t$ as $n \to \infty$. If $M > 0$, then $\phi_n(t) \to t$ uniformly on $[0, M]$, because whenever $n > M$ then, for all $t \in [0, M] \subseteq [0, n)$ we have, $|\phi_n(t) - t| = t - \phi_n(t) \leq 2^{-n}$.

Set $s_n(x) = \phi_n f(x)$. Then, s_n is a simple function. It is measurable and $s_n(x) \to f(x)$ for all $x \in X$. Furthermore, if $M \in R$ such that $\forall x \in A$ we have $f(x) \leq M$, then

$$\sup_{x \in A} | s_n(x) - f(x) | \leq \sup_{t \in [0,M]} | \varphi_n(t) - t |$$

$$\leq 2^{-n} \to 0 \text{ as } M < n \to \infty$$

10.3.3 Measurable Functions as nearly Continuous Functions

Continuity and Derivability of Functions Defined by Means of Integrals

If $f \in \mathfrak{R}[a,b]$, then function F on $[a, b]$ given by

$$F(x) = \int_a^x f$$

is welldefined, becausefor each $x \in [a,b]$,

defined on $[a, b]$

$f \in \mathfrak{R}[a, x]$ and as such $F(x)$ is uniquely

We now proceed to examine certain properties of this function F, defined on $[a, b]$.

Continuity of F and its Derivability

Theorem 10.15: If

$f \in \mathfrak{R}[a,b]$

then the function F defined on $[a, b]$ by

$F(x) = \int_a^x f(t)\, dt \ \forall\ x \in [a,b]$ is continuous on $[a, b]$ and if f is continuous at a point c of *[a, b]*, then F is derivable at c and *F'(c) = f(c).*

Proof: Continuity: If $f \in \mathfrak{R}\ [a, b]$ then f is bounded on $[a, b]$ and so is $|f|$. Let M be the upper bound of $|f|$ on $[a, b]$. For $\varepsilon > 0$, $\exists\ \delta > 0$ such that $0 < M\delta < \varepsilon$, and if $x \in [a,b]$, $x + h \in [a,b]$ and $|h| < \delta$, then

$$|F(x + h) - F(x)| = \left| \int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right|$$

$$= \left| \int_a^{x+h} f(t)dt + \int_x^a f(t)dt \right|$$

$$= \left| \int_x^{x+h} f(t)dt \right| \le \left| \int_x^{x+h} |f(t)|\, dt \right|$$

$$= \le M\,|h| < M\,\delta < \varepsilon$$

Hence, F is continuous on $[a, b]$

Derivability: For f continuous at $c \in [a,b]$, given $\varepsilon > 0\ \exists\ \delta > 0$ such that

$$|x - c| < \delta \Rightarrow |f(x) - f(c)| > \delta$$

and for $s, t \in [a, b]$, $s \neq t$, $c - \delta < s \le c \le t < c + \delta$,

$$\left| \frac{F(t) - F(s)}{t - s} - f(c) \right| = \left| \frac{1}{t - s} \int_s^t \{f(x) - f(c)dx\} \right|$$

$$\le \frac{1}{t - s} \int_s^t |f(x) - f(c)|\, dx < \varepsilon,$$

$t - s$ ^{s}

i.e., $F'(c) = f(c)$.

Corollary: If $f \in \Re\,[a, b]$ then $F(x)$ is continuous on $[a, b]$ and if f is also continuous on $[a, b]$ then $F(x)$ is derivable and $F'(x) = f(x)$ on $[a, b]$.

The above theorem asserts that a continuous function is the derivative of its integral. For this very reason the process of integration is viewed as an inverse operation of differentiation. At the same time it reflects that the process of differentiation may be viewed as the inverse operation of integration.

A derivable function f, if it exists on a domain D, such that its derivative F' equals to a given function f on D, is called a primitive of f on D. The knowledge of the primitives helps to evaluate the integral $\int_s^b f$.

Example 1: For $\sin^{-1} x$, which denotes the inverse of the functions $\sin x$ in $[0, \pi/2]$, note that

$(\sin^{-1} x)' =$

$\int$ x dt

Hence, $_0$

1

$\sqrt{1\ \ -}$

$= \sin^{-1} x, \forall\, x \in [0, 1]$ $\sqrt{1-t^2}$

(This gives another way of introducing the trigonometrical functions, through $\sin x$ defined as the inverse function of $\sin^{-1} x$ and $\sin^{-1} 1 = \pi/2$)

Besides continuity and derivability of the functions defined by means of integrals we can examine various other properties, such as uniform convergence of functional sequences defined by means of integrals.

Example 2: The sequence

$a > 0$.

x t dt

$_0 1+n t$ $\int$ —— 2

converges uniformly to 0 on [0, *a*] where

Solution: Since $\forall\ x \in [0, a],\ a > 0,$

x $0 \leq$ t

$dt \leq$

$x\ 1\ dt < a$

$\rightarrow 0$ as $n \rightarrow \infty,$

$\int_0 1+n^2 t$

$\int_0 n^2$ n^2

therefore, for $\varepsilon > 0\ \exists\ m \in$ N such that

$\int^x \frac{t}{} dt$

$< \varepsilon\ \forall\ n \geq m$ $>$

and $\forall\ x \in [0, a]$.

$_0 1+n^2 t$

x

Hence,

$\sqrt{\frac{a}{\varepsilon}}$

t dt

$\exists$ $\exists$

converges uniformly to 0, on [0, *a*] where $a > 0$.

$$\int_0 1+n^2t$$

Now, in view of the fundamental theorem, if G be any other function such that $G' = f$ besides $F' = f$ on (a, b), then $F' = G'$ gives that

$(F - G)' = 0$

i.e., $F = G$ is a constant function on (a, b).

Hence, every continuous function f admits primitives G which differ from the function F only by an additive constant. Therefore, if we can find, by any means, a primitive G of a continuous function f on an open interval containing c and d, then

$$\int_c^d f = F(d) - F(c),$$

$$= G(d) - G(c).$$

This provides a means to evaluate $\int_c^d f$

interval containing c and d.

when f is continuous on any open

Note: An independent approach to define the exponential function e^x is to consider it as the unique solution y of the equation.

$$x = \int_1^y \frac{dt}{t}, \text{ for } y > 0$$

Unless this solution is identified with e^x let us denote it by exp (x). Clearly exp (x) is non negative monotonically increasing on **R** and as $x \to -\infty$, exp

(x) $\to +0$ and as $x \to +\infty$ and exp (0) = 1. It follows that

$$\frac{dx}{dy} = \frac{1}{y}, \text{ or } \frac{dy}{dx} = y,$$

i.e., $(\exp(x))' = \exp(x)\ \forall\ x \in \mathbf{R}$.

Thus, the derivative of exp ($-x$). exp ($x + y$) with respect to x reduces to zero. So that exp (– x). exp ($x + y$) = exp (y), and on replacing x by – x and y by $x + y$ it gives

have

by

exp (x) exp (y) = exp ($x + y$), $\forall\, x, y \in \mathbf{R}$.

As the Taylor's series for exp (x) is same as for $e^x\ \forall\, x \in \mathbf{R}$, therefore, we

exp (x) = $e^x\ \forall\, x \in \mathbf{R}$.

Obviouslytheinverse functionof e^x, i.e., thenaturallogarithmlog x is defined

$$\log x = \int_1^x \frac{dt}{t},\ \forall\, x > 0.$$

Various other results concerning real logarithm and exponential functions are now simple consequences of the above analysis.

Theorem 10.16: If f has continuous derivative on (c, d) and a, $b \in (c, d)$, then

$$\int_a^b f' = f(b) - f(a).$$

Proof: Let $P \in \square\,[a,b]$. Then by mean value theorem on every $\delta_r \,\exists\, \xi_r \in (x_{r-1}, x_r)$ such that

$$f(x_r) - f(x_{r-1}) = f'(\xi_r)\,\delta_r$$

This gives,

$$\sum_1^n f'(\xi_r)\delta_r$$

$$= \sum_1^n \{ f(x_r) - f(x_{r-1})\}$$

1

$= f(b) - f(a)$.

Since $f' \in \Re[a, b]$, therefore, on letting $||P|| \to 0$ the result follows.

Corollary 1: If f is such that f' exists on $[a, b]$ and $f' \in \Re[a, b]$, then

$$\int_a^b f' = f(b) - f(a)$$

The proof is same as that of the above theorem.

Corollary 2: If f is continuous on $[a, b]$ and f' exists and is bounded and continuous on (a, b), then

$$\int_a^b f' = f(b) - f(a)$$

Proof: From the theorem, for every $c, d \in (a, b)$,

$$\int_c^d f' = f(d) - f(c)$$

When f is continuous on $[a, b]$ and $c, d \in [a, b]$ the limit of the right hand side expression in Equation (4.12) exists as $c \to a$ and $d \to b$ from above or below as the case may be; and so, also the limit of the left hand side expression exists. Hence the result.

If $a_1, \ldots, a_p$ are p points of discontinuity of f on (a, b), then on applying Corollary 2 to $[a, a_1]$, $[a_1, a_2], \ldots, [a_p, b]$, we get an extension as

Corollary 3: If f is continuous on $[a, b]$ and f' exists and is bounded and continuous on (a, b) except at a finite set of points, then

$$\int_a^b f' = f(b) - f(a).$$

10.4 EGOROFF'S THEOREM

Theorem 10.17 (Egoroff): Let (x, μ) be a measure space of finite measure, and $f_n : X \to R$ be a sequence of measurable functions convergent almost everywhere to f. Then given any $\varepsilon > 0$, there exists a measurable subset $A \subseteq X$ such that $\mu(X \setminus A) < \varepsilon$ and the sequence f_n converges uniformly to f on A.

Proof: First define

$$B_{n,m} = \bigcap_{k=n}^{\infty} \left[| f_k - f | < \frac{1}{m} \right].$$

Fix m. For most $x \in X$, $f_n(x)$ converges to $f(x)$, so there exists n such that, $| f_k(x) - f(x) | < 1/m$ for all $k \geq n$, so $x \in B_{n,m}$. Thus, we see $\{B_{n,m}\}_n \to X \setminus C$, C being some set of measure zero.

We construct the set A inductively as follows. Set $A_0 = X \setminus C$. For each $m > 0$, since $\{A_{m-1} \cap B_{n,m}\} \to A_{m-1}$, we have $\mu(A_{m-1} \setminus B_{n,m}) \to 0$, so we can choose $n(m)$ such that

$$\mu(A_{m-1} \setminus B_{n(m),m}) < \frac{\varepsilon}{2^m}$$

Furthermore set,

$A_m = A_{m-1}$

$\cap B_{n(m),m}$

Since $A_m \oplus (A_{m-1} \setminus B_{n(m),m}) = A_{m-1}$, we have

$\mu(A_m) > \mu(A_{m-1}) - \frac{\varepsilon}{}$

$> \mu(X) - \frac{\varepsilon}{2} - \frac{\varepsilon}{4} - \cdots - \frac{\varepsilon}{2^m} \geq \mu(X) - \varepsilon$

The sets A_m are decreasing, so letting

$A = \bigcap_{m=1}^{\infty} A_m = \bigcap_{m=1}^{\infty} B_{n(m),m}$

We have, $\mu(A) \geq \mu(X) - \varepsilon$, or $\mu(X \setminus A) \leq \varepsilon$. Finally, for $x \in A$, $x \in B_{n(m)}$, for all m, implies that $|f(x) - f_k(x)| < 1/m$ whenever $k \geq n(m)$. This condition is uniform for all $x \in A$.

10.5 LUSIN'S THEOREM

Egoroff's theorem says that on a set of finite measure, almost everywhere convergence of measurable functions to a finite limit is uniform convergence of a set of small measure. Lusin's theorem is a consequence of Egoroff's theorem and says that on a set of finite measure, any finite measurable function f can be restricted to a compact set K of almost full measure to form a continuous function.

Lemma: Let $A \subseteq R$ be a measurable set with $m(A) < +\infty$ and $\varepsilon > 0$. Then there is compact set $K \subseteq A$ with $m(A \setminus k) < \varepsilon$.

Proof: We know that, there is a closed subset F of A with $m(A \setminus F) < \varepsilon/2$. Since the sequence,

$F \cap [-n, n] \to F$ and $m(F) < +\infty$, there is an n_0 such that

$m(F \setminus [-n_0, n_0]) < \varepsilon/2$. The desired compact set is $F \cap [-n_0, n_0]$.

Theorem 10.18 (Lusin): Fix a measurable set $A \subseteq R$ with $m(A) < +\infty$, and let f be a real valued measurable function with domain A. For any $\varepsilon > 0$, there is a compact set $K \subseteq R$ with $m(A \setminus k) < \varepsilon$ such that the restriction of f to K is continuous.

Proof: Let (V_n) be an enumeration of the open intervals with rational endpoints in

R. Fix compact sets $K_n \subseteq f^{-1}[V_n]$ and K'_n

$\subseteq A \setminus$

$f^{-1}[V_n]$ for each n so that

$m(A \setminus (K_n \cup K'_n)) < \varepsilon / 2^n$. Now, for

$K := \bigcap_n$

$(K_n \cup K'_n)$, $m(A \setminus K) < \varepsilon$. Given x

$\in K$ and an n with $f(x) \in V_n$, $x \in 0 := K'_n$

and

$f[O \cap K] \subseteq V_n$.

The above result is true in general settings. The domain of f should have the property that sets of finite measure can be approximatedfromtheinsidebycompact sets, and for the range, there should be a countable collection of open sets V_n such

that for each open set O and each $y \in O$ there is an n with $y \in V_n \subseteq O$. This is

known as the second Axiom of countability.

Corollary 1: Let A be a measurable set such that $m(A) < \infty$. Let $f : A \to R$ be measurable function and $\varepsilon > 0$. Then there exists a step function $h : R \to R$ such that,

$m(|f - h| \geq \varepsilon) < \varepsilon$

Furthermore, if f is bounded then sup $|h| \leq$ sup $|f|$.

Proof: Let K be suchthat $f \mid K$ is continuousand $m(A\kappa) < \varepsilon$. As K is compact, we know that K is bounded, say $K \subset [-N, N]$. Since $f\ K$ is continuous, it is also uniform continuous. Thus, we may find $0 < \delta < \varepsilon$ such that

$t, s \in K$ and $| t - s | < \delta \Rightarrow | f(t) - f(s) | < \varepsilon$

Let $n > \delta^{-1}$ and $x_i = -N + \frac{i}{n}$, $i = 0, \ldots, 2Nn$. Let S be the collection of indices such that there exists $i \in K$ such that $[x_i, x_{i+1}) \cap K \neq \emptyset$. For such $i \in S$ we

may choose $y_i \in [x_i, x_{i+1}]$. We define the step function,

$$h = \sum_{i \in S} f(y_i) 1_{[x_i, x_{i+1})}$$

Let $s \in K$. Choose $i = 0, \ldots, 2Nm$ such that, $x_i \le s \le x_{i+1}$. Then

$K \cap [x_i, x_{i+1}) \cap K \neq 0/$ and $|y_i - S| < \frac{1}{n} < \delta$. We get,

$|h(s) - f(s)| = |f(y_i) - f(s)| < \varepsilon$.

Thus,

$m(|h - f| \ge \varepsilon) \le m(A \setminus K) < \varepsilon$.

Since h is constructed using the elements $f(y_i)$ we also get,

$\sup_{x \in \mathbb{L}} | h(x) | \le \sup_{x \in K} | f(x) |$

This implies the second assertion.

Corollary 2: Let $A \subset R$ be a measurable set, $f : A \to R$ be a measurable function and $\varepsilon > 0$. Then there exists a continuous function h such that,

$m(|f - h| > \varepsilon) < \varepsilon$

Moreover, we can choose h such that $\sup | h | \le \sup | f | pl$

Proof: It suffices to show that for every simple function $f = \sum_{i=1}^{m} r_i 1 \ x)$

we can find a continuous h with $\mu(|f-h|>\varepsilon)<\varepsilon$ and $|h|\le|f|$.

It can be easily shown by induction that,

$$\mu(|(\sum_i f_i)-(\sum_i h_i)|>\sum_i \varepsilon_i)\le\sum_i \mu(|f_i-h_i|>\varepsilon_i.$$

Therefore, it is sufficient to consider $f_i=1_{[x_i,\,x_{i+1}]}$. Let, $0<2\delta<x_{i+1}-x_i$. We define,

$$h_{i,\delta}(t)=\begin{cases}\delta^{-1}(t-x_i) & \text{if } x_i<t\le x_i+\delta\\ 1 & \text{if } x_i+\delta\le t\le x_{i+1}-\delta\\ \delta^{-1}(x_{i+1}-t) & \text{if } x_{i+1}-\delta\le t\le x_{i+1}\\ 0 & \text{else}\end{cases}$$

Note that $h_{i,\delta}\le 1_{(x_i,x_{i+1})}$ is continuous and that,

$$m(|h_{i,\delta}-1_{[x_i,x_{i+1})}|>0)<2\delta$$

Let, δ be such that $\frac{2\delta}{m}<\min_i(x_{i+1}-xi)$. Then, we maydefine

$$h=\sum_i r_i h_{i,\frac{\delta}{m}}$$

Hence, we have

$$m(|f-h|>\delta\le\sum_{i=1}^{m} m(r_i|1_{[x_i,x_{i+1})}-h_{i,\frac{\delta}{m}}|>\frac{\delta}{m})<2m\frac{\delta}{m}<2\delta$$

For the second assertion, we note that $|h|\le|f|$. Therefore, we also control the sup-norm.

Check Your Progress

1. Define measurable space.
2. What is step function?
3. Define a simple function.
4. What is a primitive?
5. State Egoroff's theorem.
6. 6. State Lusin's theorem.

10.6 LET US SUM UP

In this unit, you have learned that:

- Let X be a set and U be a σ-algebra on X. The pair (X, U) is called a measurable space.
- The function f is called measurable if the set $\{x: f(x) > a\}$ is measurable for every real a.
- A necessary and sufficient condition for measurability is that, $\{x \mid a \le f(x) \le b\}$ should be measurable for all a, b [including the case $a = -\infty$, $b = +\infty$], as any set of this form can be written as the intersection of two sets, $\{x \mid f(x) \ge a\} \cap \{x \mid f(x) \le b\}$.
- As (α, ∞) is an open set, we can define a measurable function as a function f defined on a measurable set E for which for every open set G in the real number system, $f^{-1}(G)$ is a measurable set.
- For any real c and two measurable real valued functions f and g, the four functions $f + c$, cf, $f + g$ and fg are measurable.
- If f is measurable, then $|f|$ is also measurable. If f and g are two measurable functions, then $f \vee g$ and $f \wedge g$ are measurable.

- If f is a measurable function on the set E and $E_1 \subset E$ is measurable set, then f is a measurable function on E_1.
- If f is a measurable function on each of the sets in a countable collection $\{E_i\}$ of disjoint measurable sets, then f is measurable.
- If a function f is continuous almost everywhere in E, then it is measurable.
- Every simple function ϕ on E is a linear combination of characteristic functions of measurable subsets of E.
- If $X = A \cup B$ and $A \cap B = \phi$, then $s = 1 = \chi + \chi$ is measurable irrespective of whether or not A and B are measurable. So if $A, B \neq \phi$, then $A, B \neq s^{-1}(\{y\})$ for any y in the range of s. Therefore, in the canonical representation we have $s(x) = \sum a_i \chi E_i(x)$, where the a_i are mutually distinct and the E_i are mutually disjoint.
- A derivable function f, if exists on a domain D, such that its derivative F equals to a given function f on D, is called a primitive of f on D.
- Let (x, μ) be a measure space of finite measure, and $f_n : X \to R$ be a

sequence of measurable functions convergent almost everywhere to f. Then given any $\varepsilon > 0$, there exists a measurable subset $A \subseteq X$ such that $\mu(X \setminus A) < \varepsilon$ and the sequence f_n converges uniformly to f on A.

- Fix a measurable set $A \subseteq R$ with $m(A) < +\infty$, and let f be a real valued measurable function with domain A. For any $\varepsilon > 0$, there is a compact set $K \subseteq R$ with $m(A \setminus k) < \varepsilon$ such that the restriction of f to K is continuous.
- A sequence $< f_n >$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0$, there is an N such that for all $n \geq N$ we have $m\{x \mid f(x) - f_n(x) | \geq \in\} < \in$.
- A sequence $\{f_n\}$ of almost everywhere finite valued measurable functions is said to be fundamental in measure, if for every $\varepsilon > 0$, $m(\{x : | f_n(x) - f_m(x) | \geq \in\}) \to 0$ as n and $m \to \infty$.

10.7 KEY WORDS

- **Measurable space:** Let X be a set and U be a σ-algebra on X. The pair (X, U) is called a measurable space
- **Step function:** A function is said to be a step function if, $f(x) = C_i$, $\xi_{i-1} < x < \xi_i$ for some subdivision of $[a, b]$ and some constants C_i
- **Finite set:** A function $s : X \to Y$ is a simple function if the range of s is a finite set.

10.8 QUESTIONS FOR REVIEW

1. List the equivalent formulations of measurable functions.
2. State the properties of measurable functions.
3. How can we approximate measurable functions by sequence of simple functions?
4. Define measurable functions as nearly continuous functions.
5. State Egoroff's theorem.

6. What is the significance of Lusin's theorem?
7. Explain the concept of measurable functions and their equivalent formulations.
8. Discuss the properties of measurable functions.
9. Describe approximation of measurable functions by sequence of simple functions.
10. Interpret measurable functions as nearly continuous functions.
11. Prove Egoroff's theorem and Lusin's theorem.

10.9 SUGGESTED READINGS AND REFERENCES

Rudin, Walter. 1976. Principles of Mathematics Analysis, 3rd edition. New York: McGraw Hill.

Carothers, N. L. 2000. *Real Analysis*, 1st edition. UK: Cambridge University Press.

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd edition. London: McGraw- Hill Education– Europe.

Barra, G.De. 1987. *Measure Theory and Integration*. New Delhi: Wiley Eastern Ltd.

Royden, H. L. 1988. *Real Analysis*, 3rd edition. New York: Macmillan Publishing Company.

Malik, S. C. and Savita Arora. 1991. *Mathematical Analysis*. New Delhi: Wiley Eastern Limited.

Gupta, S. L. and Nisha Rani. 2003. *Fundamental Real Analysis*, 4th edition.

New Delhi: Vikas Publishing House Pvt. Ltd

10.10 ANSWERS TO CHECK YOUR PROGRESS

1. Suppose X be a set and U be a σ-algebra on X. The pair (X, U) is called a measurable space.
2. A function is said to be a step function if, $f(x) = C$,

   ξ $< x < \xi$ for some

i $i-1$ i

subdivision of $[a, b]$ and some constants C_i. Clearly, a step function is a simple function.

3. A function $s : X \to Y$ is a simple function if the range of s is a finite set.
4. A derivable function f, if it exists on a domain D, such that its derivative F' equals to a given function f on D, is called a primitive of f on D.
5. Let (x, μ) be a measure space of finite measure, and $f_n : X \to R$ be a sequence of measurable functions convergent almost everywhere to f. Then given any $\varepsilon > 0$, there exists a measurable subset $A \subseteq X$ such that $\mu(X \setminus A) < \varepsilon$ and the sequence f_n converges uniformly to f on A.
6. Fix a measurable set $A \subseteq R$ with $m(A) < +\infty$, and let f be a real valued measurable function with domain A. For any $\varepsilon > 0$, there is a compact set $K \subseteq R$ with $m(A \setminus k) < \varepsilon$ such that the restriction of f to K is continuous.

UNIT 11 CONVERGENCE THEOREMS ON MEASURABLE FUNCTIONS

STRUCTURE

11.1 OBJECTIVES

After going through this unit, you will be able to:

- Explainconvergencetheorem
- Discussconvergencetheorem on measurable functions

11.2 INTRODUCTION

Convergence, in mathematics, property (exhibited by certain infinite series and functions) of approaching a limit more and more closelyas an argument (variable) of thefunction increases or decreases or as the numberof termsof the seriesincreases.Forexample,thefunction y= 1/x convergestozero as x increases. Althoughno finitevalue of *x* will causethevalue of *y* to actually become zero, the limiting value of y is zero because *y* can be made as small as desired by choosing x large enough. The line y = 0 (the x-axis) is called an asymptote of the function.

A measurable function is a function between two measurable spaces such that the preimage of any measurable set is measurable, analogouslyto the definition that a function between topological spaces is continuous if the preimage of eachopenset isopen.In realanalysis,measurablefunctionsareused in the definition of the Lebesgue integral. In probability theory, a measurable function on a probabilityspace is known as a random variable. Measurablefunctionsin measure theoryare analogousto continuous functions in topology. Acontinuous function pulls back open sets to open sets, while a measurablefunctionpullsbackmeasurable sets to measurable sets.

In this unit, you will study about the convergence theorem on measurable functions in detail.

11.3 CONVERGENCE THEOREMS ON MEASURABLE FUNCTIONS

Definition: Asequence $<f>$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0$, there is an N such that for all $n \geq N$ we have $m\{x \mid f(x) - f_n(x) \mid \geq \varepsilon\} < \varepsilon$.

Theorem 11.1 F. Riesz: Let $<f_n>$ be a sequence of measurable functions that converges in measure to f. Then there is a subsequence $<f_{n_k}>$ which converges to f almost everywhere.

Proof: Since $<f_n>$ is a sequence of measurable functions which converges in measure to f, for anypositive integer k there is an integer n_k such that for $n \geq n_k$, we have

$$m\{x \mid |f_n(x) - f(x)| \geq \frac{1}{2^k}\} < \frac{1}{2^k}$$

Let, $E_k = \{x \mid |f_{n_k}(x) - f(x)| \geq \frac{1}{2^k}\}$

Then if $x \notin \bigcup_{k=i}^{\infty} E_k$, we have

$$|f_{n_k}(x) - f(x)| < \frac{1}{2^k}$$

for $k \geq i$

and so $f_{n_k}(x) \to f(x)$.

Hence, $f_{n_k}(x) \to f(x)$ for any $x \notin A = \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} E_k$

But,

$$mA \leq m\left[\bigcup_{k=i}^{\infty} E_k\right]$$

$$= \sum_{k=i}^{\infty} mE_k$$

$$= \frac{1}{2^{k-1}}$$

Hence the measure of A is zero.

Example 11.1: Asequence $<f_n>$ which converges to zero in measure on [0,1] but such that $<f_n(x)>$ does not converge for any x in [0,1] can be constructed as

follows:

Let $n = k + 2^v$, $0 \le k \le 2^v$, and set $f_n(x) = 1$ if $x \in [k2^{-v}, (k+1)2^{-v}]$ and

$f_n(x) = 0$ otherwise. Then, $m\{x \,||\, f_n(x)| > \varepsilon\} \le \frac{2}{n}$

and so, $f_n \to 0$ in measure,

although for any $x \in [0, 1]$, the sequence $<f_n(x)>$ has the value1 for arbitrarily large values of n. So it does not converge.

Definition: A sequence $\{f_n\}$ of almost everywhere finite valued measurable functions is said to be fundamental in measure, if for every$\varepsilon > 0$,

$m(\{x : |f_n(x) - f_m(x)| \ge \varepsilon\}) \to 0$ as n and $m \to \infty$.

Definition: Asequence $\{f_n\}$ of real valuedfunctions is called fundamental almost

everywhere if there exists a set E_0 of measure zero such that, if $x \notin E_0$

and $\varepsilon > 0$,

then an integer $n_0 = n_0 = (x, \varepsilon)$ has the property that,

$|f_n(x) - f_m(x)| < \varepsilon$, whenever $n \ge n_0$ and $m \ge n_0$.

Definition: A sequence $\{f_n\}$ of almost everywhere finite valued measurable functions is said to converge to the measurable function f almost uniformly if, for every$\varepsilon > 0$, there exists ameasurable set F such that $m(F) < \varepsilon$and such that the sequence $\{f_n\}$ converges to f uniformly on F^c.

Note: Egoroff's theorem claims that on a set of finite measure,, convergence almosteverywhereimpliesalmostuniformconvergence.

Theorem 11.2: If $\{f_n\}$ is a sequence of measurable functions which converges to f almost uniformly, then $\{f_n\}$ converges to f almost everywhere.

Proof: Let F_n be a measurable set such that $m(F_n) < 1/n$ and such that the sequence $\{f_n\}$ converges to f uniformly on F_n^c, $n = 1, 2, \ldots$. If $F = \bigcap_{n=1}^{\infty} F_n$, then $m(F) \leq \mu(F_n) < \frac{1}{n}$ so that $m(F) = 0$, and it is clear that, for $x \in F^c$, $\{f_n(x)\}$ converges to $f(x)$.

11.3.1 Almost Convergence Theorem

Theorem 11.3: Almost uniform convergence implies convergence in measure.

Proof: If $\{f_n\}$ converges to f almost uniformly, then for anytwo positive numbers ε and δ there exists a measurable set F such that $m(F) < \delta$ such that $| f_n(x) - f(x) | < \varepsilon$, whenever x belongs to F^c and n is sufficiently large.

Theorem 11.4: If $\{f_n\}$ converges in measure to f, then $\{f_n\}$ is fundamental in measure. Also, if $\{f_n\}$ converges in measure to g, then $f = g$ almost everywhere.

Proof: The first claim of the theorem follows from the following relation,

$$\{x : | f_n(x) - f_m(x) | \geq \varepsilon\} \subset \{x : | f_n(x) - f(x) | \geq \tfrac{\varepsilon}{2}\} \cup \{x : | f(x) - f_m(x) | \geq \tfrac{\varepsilon}{2}\}$$

For proving the second claim, we have

$\{x : | f(x) - g(x) | \geq \varepsilon \} \subset \{x : f_n$

$(x) -$

$f(x) | \geq \frac{\varepsilon}{2}\} \cup \{x : | f_n$

$(x) - g(x) | \geq \frac{\varepsilon}{2}\}$

Since by appropriate selection of n, the measure of both sets on the right can be made arbitrarily small, we have

$m(\{x : | f(x) - g(x) | \geq \varepsilon \}) = 0$

for every $\varepsilon > 0$ which implies that $f = g$ almost everywhere.

Theorem 11.5: If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then some subsequence $\{f_n\}$ is almost uniformly fundamental.

Proof: For any positive integer k we can find an integer $n(k)$ such that if $n \geq n(k)$ and $m \geq n(k)$, then

$$m(\{x : | f_n(x) - f_m(x) | \geq \frac{1}{2^k} \}) < \frac{1}{2k}$$

We write,

$n_1 = n(1), n_2 = (n_1 + 1) \cup n(2),$

$n_3 = (n_2 + 1) \cup n(3), ...;$ then $n_1 < n_2 < n_3 < ..,$

so that the sequence $\{ f_{n_k} \}$ is certainly a subsequence of $\{k_n\}$. If,

$E = \{x : | f(x) - f(x) | \geq \frac{1}{} \}$

k n_k

n_{k+1} 2^k

and $k \le i \le j$, then for every x which does not belong to

$E_k \cup E_{k+1} \cup E_{k+2} \cup \ldots$, we have

$$| f_{n_i}(x) - f_{n_j}(x) | \le \sum_{m=i}^{\infty} | f_{n_m}(x) - f_{n_{m+1}}(x) | < \sum_{m=i}^{\infty} \frac{1}{2^m} = \frac{1}{2^{i-1}}$$

so that, in other words, the sequence $\{ f_{n_i} \}$ is uniformly fundamental on

$E \setminus (E_k \cup E_{k+1} \cup \ldots)$, since

$$m(E_k \cup E_{k+1} \cup \ldots) \le \sum_{m=k}^{\infty} m(E_m) < \frac{1}{2^{k-1}}$$

This completes the proof of the theorem.

Theorem 11.6: If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure then there exists a measurable function f such that $\{f_n\}$ converges in measure to f.

Proof:

We write

Notes

$f(x) =$ lim

$k\to\infty$

every $\varepsilon > 0$

$f_{n_k}(x)$ for every x for which the limits exists and observe that, for

$\{x : | f$

$(x) -$

$f(x) | \geq \varepsilon] \subset \{x :| f$

$(x) - f$

$(x) | \geq \frac{\varepsilon}{}\} \cup \{x :| f$

$|\geq \frac{\varepsilon}{}\}$

n n n_k 2 $n_k(x) - f(x)$ 2

Note here that, the measure of the first term on the right hand side is by hypothesis arbitrarily small if n and n_k are sufficiently large. Also, the measure of the second term also approaches 0 (as $k\to\infty$), since almost uniform convergence impliesconvergence in measure. Hence, thetheorem follows.

Note: Convergence in measure does not essentially imply pointwise convergence at anypoint.

11.3.2 Bounded Convergence Theorem

Theorem 11.7 Lebesgue bounded convergence theorem: Let $<f_n>$ be a sequenceofmeasurablefunctionsdefined on aset E offinitemeasureand suppose that $<f_n>$ is uniformlybounded, that is, there exists a real number M such that $| f_n(x) | \leq M$, for all $n \in N$ and all $x \in E$.

$\lim f(x) =$

$n\to\infty$

$f(x)$ for each X in E, then

$\int_E f = \lim_{n\to\infty} \int_E f_n$.

Proof: We will apply Egoroff's theorem to prove this theorem.

Therefore, for a

given ε > 0, there is an N and a measurable set $E_0 \subset E$ such that $mE_0^c <$

ε/4,

and for $n \geq M$ and $x \in E_0$

$| f_n (x) -$

Then we have,

we have

$f(x) | < \frac{\varepsilon}{2m(E)}$

$$\left| \int_E f_n - \int_E f \right| = \left| \int_E (f_n - f) \right| \leq \int_E | f_n - f |$$

$$= \int_{E_0} | f_n - f | + \int_{E_0^c} | f_n - f |$$

Hence,

$$< \frac{\varepsilon}{2m(E)} . m(E_0) + \frac{\varepsilon}{4M} . 2M$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

$$\int_E f_n \to \int_E f$$

Thus, the theorem is proved.

11.3.3 Lebesgue Convergence Theorem

Theorem 11.8 Lebesgue's dominated convergence theorem: Let $A \in$ A

,

(f_n) be a sequence of measurable functions such that $f_n(x) \to f(x)$ $(x \in A)$. If there

exists a function $g \in L^1(\mu)$ on A such that,

$|f_n(x)| \leq g(x)$

then,

$\lim_n \int_A f_n d\mu = \int_A f d\mu$.

Proof: From $|f_n(x)| \leq g(x)$ we get $f_n \in L^1(\mu)$. As $f_n + g \geq 0$ and $f + g \geq 0$, by

Fatou's lemma it follows that,

$\int_A (f + g) d\mu \leq \underline{\lim}_n \int_A (f_n + g)$

or,

$\int_A f d\mu \leq \underline{\lim}_n \int_A f_n d\mu$

Since $g - f \geq 0$, in the same way

$\int_A (g - f) d\mu \leq \underline{\lim}_n \int_A (g - f_n) d\mu$

So that,

$-\int_A f d\mu \leq -\underline{\lim}_n \int_A f_n d\mu$

which is the same as

$\int_A f d\mu \geq \overline{\lim}_n \int_A f_n d\mu$ —

Hence,

$\underline{\lim}_n \int_A f_n d\mu = \overline{\lim}_n \int_A f_n d\mu = \int_A f d\mu$

Check Your Progress

1. When does a sequence <fn> of measurable functions is said to converge?
2. State almost convergence theorem.
3. State Lebesgue bounded convergence theorem.
4. State Lebesgue's criterion for integrability.
5. State monotone convergence theorem.
6. Write the condition for a measurable function to be integrable.
7. State Lebesgue's dominated convergence theorem.

11.4 LET US SUM UP

- A sequence $<f_n>$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0$, there is an N such that for all $n \geq N$ we have $m\{x \mid f(x) - f_n(x) \mid \geq \varepsilon\} < \varepsilon$.
- Let $<f_n>$ be a sequence of measurable functions that converges in measure to f. Then there is a subsequence $<f_{n_k}>$ which converges to f almost everywhere.
- Since $<f_n>$ is a sequence of measurable functions which converges in measure to f, for any positive integer k there is an integer n_k such that for $n \geq n_k$, we have $m\{x \mid f_n(x) - f(x) \mid \geq \frac{1}{2^k}\} < \frac{1}{2^k}$
- A sequence $<f_n>$ which converges to zero in measure on [0,1] but such that $<f_n(x)>$ does not converge for any x in [0,1] can be constructed as follows: Let $n = k + 2^v$, $0 \leq k \leq 2^v$, and set $f_n(x) = 1$ if $x \in [k2^{-v}, (k+1)2^{-v}]$ and

$f_n(x) = 0$ otherwise.

- A sequence $\{f_n\}$ of almost everywhere finite valued measurable functions is said to be fundamental in measure, if for every $\varepsilon > 0$,

$$m(\{x : | f_n(x) - f_m(x) | \geq \varepsilon \}) \to 0 \text{ as } n \text{ and } m \to \infty.$$

- A sequence $\{f_n\}$ of real valued functions is called fundamental almost everywhere if there exists a set E_0 of measure zero such that, if $x \notin E_0$ and $\varepsilon > 0$, then an integer $n_0 = n_0(x, \varepsilon)$ has the property that,

$| f_n(x) - f_m(x) | < \varepsilon$, whenever $n \geq n_0$ and $m \geq n_0$.

- A sequence $\{f_n\}$ of almost everywhere finite valued measurable functions is said to converge to the measurable function f almost uniformly if, for every $\varepsilon > 0$, there exists a measurable set F such that $m(F) < \varepsilon$ and such that the sequence $\{f_n\}$ converges to f uniformly on F^c.
- Egoroff's theorem claims that on a set of finite measure,, convergence almost everywhere implies almost uniform convergence.
- If $\{f_n\}$ is a sequence of measurable functions which converges to f almost uniformly, then $\{f_n\}$ converges to f almost everywhere.
- Let F_n be a measurable set such that $m(F_n) < 1/n$ and such that the sequence $\{f_n\}$ converges to f uniformly on F_n^c, $n = 1, 2, \ldots$. If $F = \bigcap_{n=1}^{\infty} F_n$, then $m(F) \leq \mu(F_n) < \frac{1}{n}$ so that $m(F) = 0$, and it is clear that, for $x \in F^c$, $\{f_n(x)\}$ converges to $f(x)$.
- Almost uniform convergence implies convergence in measure.
- If $\{f_n\}$ converges to f almost uniformly, then for any two positive numbers ε and δ there exists a measurable set F such that $m(F) < \delta$ such that $| f_n(x) - f(x) | < \varepsilon$, whenever x belongs to F^c and n is sufficiently large.
- If $\{f_n\}$ converges in measure to f, then $\{f_n\}$ is fundamental in measure. Also,

if $\{f_n\}$ converges in measure to g, then $f = g$ almost everywhere.

- Since by appropriate selection of n, the measure of both sets on the right can be made arbitrarily small, we have

$m(\{x : | f(x) - g(x) | \geq \varepsilon \}) = 0$

for every $\varepsilon > 0$ which implies that $f = g$ almost everywhere.

- If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure, then some subsequence $\{f_n\}$ is almost uniformly fundamental.
- For any positive integer k we can find an integer $n(k)$ such that if $n \geq n(k)$ and $m \geq n(k)$, then

$$m(\{x : | f_n(x) - f_m(x) | \geq \frac{1}{2^k} \}) < \frac{1}{2k}$$

- If $\{f_n\}$ is a sequence of measurable functions which is fundamental in measure then there exists a measurable function f such that $\{f_n\}$ converges in measure to f.
- **Lebesgue bounded convergence theorem:** Let $<f_n>$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $<f_n>$ is uniformly bounded, that is, there exists a real number M such that $|f_n(x)| \leq M$, for all $n \in N$ and all $x \in E$.
- We will apply Egoroff's theorem to prove this theorem. Therefore, for a given $\varepsilon > 0$, there is an N and a measurable set $E_0 \subset E$ such that $mE_0^c < \varepsilon/4$?, and for $n \geq M$ and $x \in E_0$

we have

$| f_n(x) - f(x) | < \frac{\varepsilon}{2m(E)}$

Then we have,

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \le \int_E |f_n - f|$$

$$= \int_{E_0} |f_n - f| + \int_{E_0^c} |f_n - f|$$

$$< \frac{\varepsilon}{2m(E_0)} \, . \, m(E_0) + \frac{\varepsilon}{4M} \, . \, 2M$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence,

$$\int_E f_n \to \int_E f$$

Thus, the theorem is proved.

- Lebesgue's dominated convergence theorem: Let $A \in \mathcal{A}$, (f_n) be a sequence of measurable functions such that $f_n(x) \to f(x)$ $(x \in A)$. If there exists a function $g \in L^1(\mu)$ on A such that,

  $|f_n(x)| \le g(x)$

11.5 KEY WORDS

- **Measurable space:** Let X be a set and U be a σ-algebra on X. The pair (X, U) is called a measurable space
- **Step function:** A function is said to be a step function if, $f(x) = C_i$, $\xi_{i-1} < x < \xi_i$ for some subdivision of $[a, b]$ and some constants C_i
- **Simple function:** A function $s : X \to Y$ is a simple function if the range of s

is a finite set

11.6 QUESTION FOR REVIEW

Short Answer Questions

1. What do you understand byconvergence in measure?
2. What is the use of almost convergence theorem?
3. What is the significance of boundedconvergencetheorem?
4. Stategeneral Lebesgueintegral.
5. Write an application of Lebesgue convergencetheorem.
6. Discuss the properties of measurablefunctions.
7. Explain the convergence in measureand F. Riesz theorem for convergence in measure.
8. Illustrate almostconvergencetheorem.
9. IllustrateLebesgueintegral of aboundedfunctionoveraset of finitemeasure and itsproperties.
10. State and prove bounded convergence theorem.
11. Describe Lebesguetheorem regardingpoints of discontinuities of Riemann integrablefunctions.
12. State and prove monotone convergence theorem.

11.7 SUGGESTED READINGS AND REFERENCES

Rudin, Walter. 1976. *Principles of Mathematics Analysis*, 3rd edition. New York: McGraw Hill.

Carothers, N. L. 2000. *Real Analysis*, 1st edition. UK: Cambridge University Press.

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd edition. London: McGraw- Hill Education– Europe.

Barra, G.De. 1987. *Measure Theory and Integration*. New Delhi: WileyEastern Ltd.

Royden, H. L. 1988. *Real Analysis*, 3rdedition. New York: Macmillan

Publishing Company.

Malik, S. C. and Savita Arora. 1991. *Mathematical Analysis*. New Delhi: Wiley Eastern Limited.

Gupta, S. L. and Nisha Rani. 2003. *Fundamental Real Analysis*, 4th edition.

New Delhi: Vikas Publishing House Pvt. Ltd.

11.8 ANSWERS TO CHECK YOUR PROGRESS

QUESTIONS

1. A sequence $<f_n>$ of measurable functions is said to converge to f in measure if, given $\varepsilon > 0$, there is an N such that for all $n \geq N$ we have

$$m\{x \mid f(x) - f_n(x) \mid \geq \varepsilon\} < \varepsilon$$

2. Almost uniform convergence implies convergence in measure.
3. Let $<f_n>$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $<f_n>$ is uniformly bounded, that is, there exists a real number M such that $|f_n(x)| \leq M$, for all $n \in N$ and all $x \in E$.

If $\lim_{n\to\infty} f_n(x) = f(x)$ for each X in E, then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

4. Let $f : [a, b] \to R$. Then, f is Riemann integrable if and only if f is bounded and the set of discontinuities of f has measure 0.
5. Let (f_n) be non decreasing sequence of non negative measurable functions with limit f.

Then, $\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu$,

$A \in$ A

6. A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In this case we define,

$$\int_E f = \int_E f^+ - \int_E f^-$$

7. Let $A \in$ A, (f_n) be a sequence of measurable functions such that $f_n(x) \to f(x)$ $(x \in A)$. If there exists a function $g \in L^1(\mu)$ on A such that,

$|f_n(x)| \leq g(x)$

then, $$\lim_n \int_A f_n d\mu = \int_A f d\mu.$$

CHAPTER 12 PRODUCT MEASURES METRIC OUTER MEASURES AND HAUSDORFF MEASURE

STRUCTURE

12.1 OBJECTIVE

In this unit we will describe the details about product measures .

12.2 INTRODUCTION

In mathematics a **Hausdorff measure** is a type of outer measure, named for Felix Hausdorff, The zero-dimensional Hausdorff measure is the number of points in the set (if the set is finite) or ∞ if the set is infinite. In mathematics, in particular in measure theory, an **outer measure** or **exterior measure** is a function defined on all subsets of a given set with values in the extended real numbers satisfying some additional technical conditions. A general theory of outer measures was first introduced by Constantin Carathéodory to provide a basis for the theory of measurable sets and countably additive measures. Carathéodory's work on outer measures found many applications in measure-theoretic set theory (outer measures are for example used in the proof of the fundamental Carathéodory's extension theorem), and was used in an essential way by Hausdorff to define a dimension-like metric invariant now called Hausdorff dimension.

12.3 PRODUCT MEASURES

Definition 1.1. If X and Y are any two sets, their Cartesian product $X \times Y$ is
thesetofallorderpairs$\{(x,y): x\in X, y\in Y\}$.
If $A \subset X$, $B \subset Y$, $A\times B \subset X\times Y$ is called a rectangle. Suppose (X, A), (X, B) are measurable spaces. A measurable rectangle is a set of the form $A \times B$, $A \in A$, $B\in B$.Asetoftheform
$Q=R1 \cup ... \cup Rn$,
where the Ri are disjoint measurable rectangles, is called an elementary sets. We
denote this collection by E.
Exercise 1.1. Prove that the elementary sets form an algebra. That is, E is closed under complementation and finite unions.
We shall denote by $A\times B$ the σ–algebra generated by the measurable rectangle which is the same as the σ–algebra generated by the elementary sets.

Product of finite number of measure spaces

Let (X1, A1, μ1), . . . , (Xn, An, μn) be σ-finite measure spaces. Then $A1 \times\cdots\times An =\sigma(\{A1 \times\cdots\times An | Ai \in Ai \text{ for } i=1,...,n\})$.
Using theorem 13.0.16 (applied $n - 1$ times), we can construct a unique measure $\mu1 \times\cdots \mu n$ on $A1 \times\cdots\times An$ that satisfies
$(\mu1 \times\cdots\mu n)(A1 \times\cdots\times An)=\mu(A1)\cdots\mu(An)$
whenever $Ai \in Ai$ for $i = 1, . . . , n$. Integrals of functions with respect to $\mu1 \times \cdots \mu n$ can be evaluated by repeated applications of Fubini’s theorem

Check your progress

1. Prove that B(R) × B(R) = B(R2).

12.4 METRIC OUTER MEASURES.

Let (X,d) be a metric space. We recall that, if E,F are non-empty subsets of

X, the quantity $d(E,F) = \inf\{d(x,y) | x \in E, y \in F\}$

is *the distance between E and F*.

Definition 5.4 *Let* (X,d) *be a metric space and* μ^* *be an outer measure on* X. *We say that* μ^* *is a metric outer measure if*

$$\mu^*(E \cup F) = \mu^*(E) + \mu^*(F)$$

for every non-empty $E,F \subseteq X$ *with* $d(E,F) > 0$.

Theorem 5.8 *Let* (X,d) *be a metric space and* μ^* *an outer measure on* X. *Then, the measure* μ *which is induced by* μ^* *on* (X,Σ_{μ^*}) *is a Borel measure (i.e. all Borel sets in* X *are* μ^**-measurable) if and only if* μ^* *is a metric outer measure.*

Proof: Suppose that all Borel sets in X are μ^*-measurable and take arbitrary non-empty $E,F \subseteq X$ with $d(E,F) > 0$. We consider $r = d(E,F)$ and the open set $U = \cup_{x \in E} B(x;r)$. It is clear that $E \subseteq U$ and $F \cap U = \emptyset$. Since U is μ^*-measurable, we have $\mu^*(E \cup F) = \mu^*\big((E \cup F) \cap U\big) + \mu^*\big((E \cup F) \cap U^c\big) =$ $\mu^*(E) + \mu^*(F)$. Therefore, μ^* is a metric outer measure on X.

Now let μ^* be a metric outer measure and consider an open $U \subseteq X$. If A is a non-empty subset of U, we define

$$A_n = \left\{x \in A \,|\, d(x,y) \geq \frac{1}{n}\right. \qquad \text{o for every } y \notin U.$$

It is obvious that $A_n \subseteq A_{n+1}$ for all n. If $x \in A \subseteq U$, there is $r > 0$ so that $B(x;r) \subseteq U$ and, if we take $n \in \mathbf{N}$ so that $\frac{1}{n} \leq r$, then $x \in A_n$. Therefore,

$$A_n \uparrow A.$$

We define, now, $B_1 = A_1$ and $B_n = A_n \setminus A_{n-1}$ for all $n \geq 2$ and have that the sets $B_1, B_2, \ldots$ are pairwise disjoint and that $A = \cup_{n=1}^{+\infty} B_n$. If $x \in A_n$ and $z \in B_{n+2}$, then $z \notin A_{n+1}$ and there is some $y \notin U$ so that $d(y,z) < \frac{1}{n+1}$. Then

$d(x,z) \geq d(x,y) - d(y,z) > \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}$. Therefore,

$$d(A_n, B_{n+2}) \geq \frac{1}{n(n+1)} > 0$$

for every n. Since $A_{n+2} \supseteq A_n \cup B_{n+2}$, we find $\mu^*(A_{n+2}) \geq \mu^*(A_n \cup B_{n+2}) = \mu^*(A_n) + \mu^*(B_{n+2})$. By induction we get

$$\mu*(B1) + \mu*(B3) + \cdots + \mu*(B2n+1) \leq \mu*(A2n+1)$$

and $\mu^*(B_2) + \mu^*(B_4) + \cdots + \mu^*(B_{2n}) \leq \mu^*(A_{2n})$

for all n. If at least one of the series $\mu^*(B_1)+\mu^*(B_3)+\cdots$ and $\mu^*(B_2)+\mu^*(B_4)+\cdots$ diverges to $+\infty$, then either $\mu^*(A_{2n+1}) \to +\infty$ or $\mu^*(A_{2n}) \to +\infty$. Since the sequence $(\mu^*(A_n))$ is increasing, we get that in both cases it diverges to $+\infty$. Since, also $\mu^*(A_n) \leq \mu^*(A)$ for all n, we get that $\mu^*(A_n) \uparrow \mu^*(A)$. If both series $\mu^*(B_1)+\mu^*(B_3)+\cdots$ and $\mu^*(B_2)+\mu^*(B_4)+\cdots$ converge, for every > 0 there is n so that $\sum_{k=n+1}^{+\infty} \mu^*(B_k) < \epsilon$. Now, $\mu^*(A) \leq \mu^*(A_n) + \sum_{k=n+1}^{+\infty} \mu^*(B_k) \leq \mu^*(A_n) + \epsilon$. This implies that $\mu^*(A_n) \uparrow \mu^*(A)$. Therefore, in any case,

$$\mu^*(A_n) \uparrow \mu^*(A).$$

We consider an arbitrary $E \subseteq X$ and we take $A = E \cap U$. Since $E \cup U^c \subseteq U^c$, we have that $d(A_n, E \cap U^c) > 0$ for all n and, hence, $\mu^*(E) \geq \mu^*\big(A_n \cup (E \cap U^c)\big) = \mu^*(A_n) + \mu^*(E \cap U^c)$ for all n. Taking the limit as $n \to +\infty$, we find

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \cap U^c).$$

We conclude that every U open in X is μ^*-measurable and, hence, every Borel set in X is μ^*-measurable.

12.5 HAUSDORFF MEASURE.

Let (X,d) be a metric space. The *diameter* of a non-empty set $E \subseteq X$ is defined as $\text{diam}(E) = \sup\{d(x,y) | x,y \in E\}$ and the diameter of the $\emptyset$ is defined as $\text{diam}(\emptyset) = 0$.

We take an arbitrary $\delta > 0$ and consider the collection C_δ of all subsets of X of diameter not larger than δ. We, then, fix some α with $0 < \alpha < +\infty$ and consider the function $\tau_{\alpha,\delta} : C_\delta \to [0,+\infty]$ defined by $\tau_{\alpha,\delta}(E) = \text{diam}(E))^\alpha$ for every $E \in C_\delta$. We are, now, ready to apply Theorem 3.2 and define

$+\infty$ $h^*_{\alpha,\delta}(E)$ = inf $^{\text{nX}}$ diam$(E_j))^\alpha \mid E \subseteq \cup_{j=1}^{+\infty} E_j$,diam$(E_j) \leq \delta$ for all j^{o}.

j=1

We have that $h^*_{\alpha,\delta}$ is an outer measure on X and we further define

$$h^*_\alpha(E) = \sup h^*_{\alpha,\delta}(E), \qquad E \subseteq X.$$

δ>0

We observe that, if $0 < \delta_1 < \delta_2$, then the set whose infimum is $h^*_{\alpha,\delta}1(E)$ is included in the set whose infimum is $h^*_{\alpha,\delta}2(E)$. Therefore, $h^*_{\alpha,\delta}2(E) \leq h^*_{\alpha,\delta}1(E)$ and, hence,

$$h^*_\alpha(E) = \lim_{\delta \to 0+} h^*_{\alpha,\delta}(E), \qquad E \subseteq X.$$

Theorem 5.9 *Let* (X,d) *be a metric space and* $0 < \alpha < +\infty$*. Then,* h^*_α *is a metric outer measure on X.*

Proof: We have $h^*_\alpha(\emptyset) = \sup_{\delta>0} h^*_{\alpha,\delta}(\emptyset) = 0$, since $h^*_{\alpha,\delta}$ is an outer measure for every $\delta > 0$.

If $E \subseteq F \subseteq X$, then for every $\delta > 0$ we have $h^*_{\alpha,\delta}(E) \leq h_{\alpha,\delta}{}^*(F) \leq h^*_\alpha(F)$. Taking the supremum of the left side, we find $h^*_\alpha(E) \leq h^*_\alpha(F)$.

If$E = \cup_{j=1}^{+\infty} E_j$, then for every $\delta > 0$ we have $h^*_{\alpha,\delta}(E) \leq \sum_{j=1}^{+\infty} h^*_{\alpha,\delta}(E_j) \leq$ $^{\text{P+}}j=1^\infty$ $h^*_\alpha(E_j)$ and, taking the supremum of the left side, we find $h^*_\alpha(E) \leq$ P+j=1∞ $h{*}\alpha(Ej)$.

Therefore, h^*_α is an outer measure on X.

Now, take any $E,F \subseteq X$ with $d(E,F) > 0$. If $h^*_\alpha(E \cup F) = +\infty$, then the equality$h^*_\alpha(E \cup F) = h^*_\alpha(E) + h^*_\alpha(F)$ is clearly true. We suppose that$h^*_\alpha(E \cup F) < +\infty$ and, hence, $h^*_{\alpha,\delta}(E \cup F) < +\infty$ for every $\delta > 0$. We take arbitrary $\delta < d(E,F)$ and an arbitrary covering $E \cup F \subseteq \cup_{j=1}^{+\infty} A_j$ with diam$(A_j) \leq \delta$ for every j. It is obvious that each A_j intersects at most one of the E and F. We set $B_j = A_j$ when A_j intersects E and $B_j = \emptyset$ otherwise h^*_α and, similarly,

$C_j = A_j$ when A_j intersects F and $C_j = \emptyset$ otherwise. Then, $E \subseteq \cup_{j=1}^{+\infty} B_j$ and $F \subseteq \cup_{j=1}^{+\infty} C_j$ and, hence, $h^*_{\alpha,\delta}(E) \leq \sum_{j=1}^{+\infty}(\text{diam}(B_j))^\alpha$ and $h^*_{\alpha,\delta}(F) \leq \sum_{j=1}^{+\infty}(\text{diam}(C_j))^\alpha$. Adding, we find $h^*_{\alpha,\delta}(E) + h^*_{\alpha,\delta}(F) \leq \sum_{j=1}^{+\infty}(\text{diam}(A_j))^\alpha$ and, taking the infimum of the right side, $h^*_{\alpha,\delta}(E) + h^*_{\alpha,\delta}(F) \leq h^*_{\alpha,\delta}(E \cup F)$.

Taking the limit as $\delta \to 0+$ we find $h^*_\alpha(E) + h^*_\alpha(F) \leq h^*_\alpha(E \cup F)$ and, since the opposite inequality is obvious, we conclude that

$$h^*_\alpha(E) + h^*_\alpha(F) = h^*_\alpha(E \cup F).$$

Definition 5.5 *Let* (X,d) *be a metric space and* $0 < \alpha < +\infty$. *We call the α-dimensional Hausdorff outer measure on X and the measure* h_α *on* $(X, \Sigma_{h^*_\alpha})$ *is called the α-dimensional Hausdorff measure on X.*

Proposition 5.3 *Let* (X,d) *be a metric space, E a Borel set in X and let* $0 < \alpha_1 < \alpha_2 < +\infty$. *If* $h_{\alpha 1}(E) < +\infty$, *then* $h_{\alpha 2}(E) = 0$.

Proof: Since $h^*_{\alpha 1}(E) = h_{\alpha 1}(E) < +\infty$, we have that $h^*_{\alpha_1,\delta}(E) < +\infty$ for every $\delta > 0$. We fix such a $\delta > 0$ and consider a covering $E \subseteq \cup_{j=1}^{+\infty} A_j$ by subsets of

X with $\text{diam}(A_j) \leq \delta$ for all j so that $\sum_{j=1}^{+\infty} \text{diam}(A_j))^{\alpha_1} < h^*_{\alpha_1,\delta}(E) + 1 \leq h^*_{\alpha 1}(E) + 1$.

Therefore, $h^*_{\alpha_2,\delta}(E) \leq \sum_{j=1}^{+\infty} \text{diam}(A_j))^{\alpha_2} \leq \delta^{\alpha_2-\alpha_1} \sum_{j=1}^{+\infty} \text{diam}(A_j))^{\alpha_1} \leq (h^*_{\alpha_1}(E) + 1)\delta^{\alpha_2-\alpha_1}$ and, taking the limit as $\delta \to 0+$, we find $h^*_{\alpha_2}(E) = 0$. Hence, $h_{\alpha 2}(E) = 0$.

Proposition 5.4 *If E is any Borel set in a metric space* (X,d), *there is an* $\alpha_0 \in [0,+\infty]$ *with the property that* $h_\alpha(E) = +\infty$ *for every* $\alpha \in (0,\alpha_0)$ *and* $h_\alpha(E) = 0$ *for every* $\alpha \in (\alpha_0,+\infty)$.

Proof: We consider various cases.

$h_\alpha(E) = 0$ for every $\alpha > 0$. In this case we set $\alpha_0 = 0$.

$h_\alpha(E) = +\infty$ for every $\alpha > 0$. We, now, set $\alpha_0 = +\infty$.

There are α_1 and α_2 in $(0,+\infty)$ so that $0 < h_{\alpha 1}(E)$ and $h_{\alpha 2}(E) < +\infty$.

Proposition 5.3 implies that $\alpha_1 \leq \alpha_2$ and that $h_\alpha(E) = +\infty$ for every $\alpha \in (0,\alpha_1)$ and $h_\alpha(E) = 0$ for every $\alpha \in (\alpha_2,+\infty)$. We consider the set $\{\alpha \in (0,+\infty) | h_\alpha(E) = +\infty\}$ and its supremum $\alpha_0 \in [\alpha_1,\alpha_2]$. The same Proposition 5.3 implies that $h_\alpha(E) = +\infty$ for every $\alpha \in (0,\alpha_0)$ and $h_\alpha(E) = 0$ for every $\alpha \in (\alpha_0,+\infty)$.

Definition 5.6 *If E is any Borel set in a metric space* (X,d), *the* a_0 *of Proposition 5.4 is called the Hausdorff dimension of E and it is denoted* $\dim_h(E)$

Check your progress

2.Proove that the following

Let (X,d) be a metric space and μ^* an outer measure on X. Then, the measure μ which is induced by μ^* on (X,Σ_{μ^*}) is a Borel measure (i.e. all Borel sets in X are μ^*-measurable) if and only if μ^* is a metric outer measure.

12.6 LET US SUMUP

In this unit we discussed about product measures in detail. In this unit we discussed about the Hausdorff measure and Metric outer measures.

12.7 KEYWORDS

Outer measure

Hausdorff dimension

lebesgue measure

Product measure

Measurable space

12.8 QUESTIONS FOR REVIEW

1.Prove that the elementary sets form an algebra. That is, E is closed under complementation and finite unions.

Let (X,d) be a metric space and $0 < \alpha < +\infty$. Then

Prove that

,h^*_α is a metric outer measure on X.

12.9 SUGGESTED READINGS AND REFERENCES

Fundamentals of Real Analysis, S K. Berberian, Springer.

An introduction to measure theory Terence Tao

Measure Theory Authors: **Bogachev**, Vladimir I

Chovanec Ferdinand. Cantor sets. Sci. Military J. 2010

Christopher Shaver. An exploration of the cantor set. Rose-Hulman Undergraduate Mathematics Journal.

Dauben Joseph Warren, Corinthians I. Georg cantor: The battle for transfinite set theory. American Mathematical Society.

Su Francis E, et al. Devil's staircase. Math Fun Facts. http://www.math.hmc.edu/funfacts, http://www.math.hmc.edu/funfacts

Amir D. Aczel, A Strange Wilderness the Lives of the Great Mathematicians, Sterling Publishing Co. 2011.

Planetmath.org

Proofwiki.Org

12.10 ANSWERS TO CHECK YOUR PROGRESS

1. Please check section 12.3 for Question 1
2. check out section theorm 5.8 For answer to check your progress

UNIT 13 LEBESGUE INTEGRAL OF NONNEGATIVE MEASURABLE FUNCTION

STRUCTURE

13.1 OBJECTIVES

After going through this unit, you will be able to:

- Understandwhat Lebesgue integral is
- Explain Lebesgueintegral of non-negative measurablefunction
- Discussgeneral Lebesgueintegral

13.2 INTRODUCTION

Lebesgue integration is an alternative wayof defining the integral in terms of measure theory that is used to integratea muchbroaderclass of functions thanthe Riemann integral or even the Riemann-Stieltjes integral. The idea behind the Lebesgueintegral is that instead of approximatingthetotalareabydividingitinto vertical strips, one approximates the total area by dividing it into horizontal strips. This corresponds to asking 'for each y-value, how many x-values produce this value?' as opposed to asking 'for each x-value, what y-value does it produce?'

Because the Lebesgue integral is defined in a way that does not depend on the structure of R, it is able to integrate many functions that cannot be integrated otherwise. Furthermore, the Lebesgue integral can define the integral in a completely abstract setting, giving rise to probability theory.

In this unit you will study about Lebesgue integral, Lebesgue integral of non-negative measurable function and general Lebesgue integral.

13.3 LEBESGUE INTEGRAL OF NONNEGATIVE MEASURABLE FUNCTION

In measure theory, a measurable function is defined as a function between two measurable spaces such that the pre-image of any measurable set is measurable. Principally, in analysis, the measurable functions are the Lebesgue integral.

The Lebesgue integral of non-negative Lebesgue measurable functions define the Lebesgue integral for simple functions. In addition, when a bounded function is defined on a Lebesgue measurable set E with $m(E) < \infty$ then it is Lebesgue integrable.

Throughout this section we will be using the measure space (X, F, μ).

Definition: Let s be a non negative F measurable simple function so that,

$$s = \sum_{i=1}^{N} a_i X_{A_i}$$

with disjoint F measurable sets A_i, $\bigcup_{i=1}^{N} A_i = X$ and $a_i \geq 0$. For any

$E \in F$ define the integral of f over E to be,

N

$I_E(s) = \sum a_i \mu(A_i \cap E)$

$i = 1$

with the

convention that if $a = 0$ and $\mu(A \cap E) = +\infty$ then $0 \times (+\infty) = 0$. So the area under

i i

$s \equiv 0$ in R is zero.

Example 13.1: Consider ([0, 1], □, μ) . Define,

$f(x) = \begin{cases} 1 & \text{if} \\ 0 & \text{if} \end{cases}$

x rational

x irrational

This is a simple function with $A = Q \cap [0,1] \in L$ and A the set of irrationals

1 0

in [0,1] which, as the complement of □$_1$, is in L. Thus, f is measurable and

$$I_{[0,1]}(f) = 1\mu(□ \cap [0, 1]) + 0\mu(□^C \cap [0, 1])$$

$$= 0$$

since, the Lebesgue measure of a countable set is zero.

Lemma 1: If $E_1 \subseteq E$ 2

$\subseteq E$ … are in F and $E = □^{\infty}$

E_n then,

3 $n=$

$\lim \mu(E\ ) = \mu(E)$ n

$n \to \infty$

and we say that we have an increasing sequence of sets.

Proof: If there exists an n such that $\mu(E\ ) = +\infty$ then $E \subseteq E$ implies $\mu(E)$

$= +\infty$

n n

and the result follows.

∞

So assume that $\mu(E\ ) < +\infty$ for all $n \geq 1$. Then, $E = E \cup \square \left(E_n \setminus E_{n-1}\right)$ is a

n

disjoint union. Note that E

$\subseteq E$ implies that $E = (E \setminus E$

1 $n=2$

$) \cup E$

, which is a

$n-1$ n

n n $n-1$

$n-1$

disjoint union. So $\mu(E\ ) = \mu(E \setminus E$

$) + \mu(E$

). Because the measures are finite,

n

we can rearrange this as

$n\ n-1$

$n-1$

$\mu(E \setminus E$

$) = \mu(E\ ) - \mu(E$

). So,

$n\ n-1$

n $n-1$

∞

$\mu(E) = \mu(E\ ) + \sum \mu(E_n \setminus E_{n-1})$

1 $n=2$

N

$\mu(E\ ) + \lim \sum$

$(\mu(E_n) - \mu(E_{n-1}))$

$=$ $\lim_{N\to\infty} \sum_{n=1}^{N}$

(Bythe definition of infinite sum)

$\lim_{N\to\infty} \mu(E_N) =$

Theorem 13.1: Let s and t be two simple non negative F measurable functions on (X, F, μ) and $E, F \in F$. Then,

1. $I_E(cs) = cI_E(s)$ for all $c \in R$. 2. $I_E(s + t) = I_E(s) + I_E(t)$.

3. If $s \leq t$ on E then $I_E(s) \leq I_E(t)$. If $F \subseteq E$ then $I_F(s) \leq I_E(s)$.

4. If $E_1 \subseteq E_2 \subseteq E_3 \subseteq \ldots$ and $E = \bigcup_{k=1}^{\infty} E_k$ then $\lim_{k\to\infty} I_{E_k}(s) = I_E(s)$.

Proof: As in the Lemma above write,

and

$$s = \sum_{i=1}^{M} a_i X_{A_i}$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{N} a_i XC_{ij}$$

$$t = \sum_{j=1}^{N} b_j XB_j$$

$$= \sum^{M}\sum^{N} b_j XC_{ij}$$

with $C_{ij} = A_i \cap B_j \in F$.

1. Note that $cs = \sum_{i=1}^{M} ca_i X A_i$

and so,

$$I_E(cs) = \sum_{i=1}^{M} ca_i \mu(A_i)$$

$$= c\sum_{i=1}^{M} a_i \mu(A_i) = cI_E(s)$$

2. Then $s + t = \sum_{i=1}^{M} \sum_{j=1}^{N} (a_i + b_j) XC_{ij}$. So,

$$I_E(s+t) = \sum_{i=1}^{M}\sum_{j=1}^{N}(a_i + b_j)\mu(C_{ij} \cap E)$$

$$= \sum_{i=1}^{M}\sum_{j=1}^{N} a_i \mu(C_{ij} \cap E) + \sum_{i=1}^{M}\sum_{j=1}^{N} b_i \mu(C_{ij} \cap E)$$

$$= \sum_{i=1}^{M} a_i \mu\left(\bigcup_{j=1}^{N} (C_{ij} \cap E)\right) + \sum_{j=1}^{N} b_i \mu\left(\bigcup_{i=1}^{M} (C_{ij} \cap E)\right)$$

$$= \sum_{i=1}^{M} a_i \mu(A_i \cap E) + \sum_{j=1}^{N} b_j \mu(B_j \cap E)$$

$$= I_E(s) + I_E(t)$$

3. Given any $1 \le i \le M$, $1 \le j \le N$ for which $C_i \cap E \neq \phi$ we have for any $x \in C_i \cap E$ that

$$a_i = s(x) \le t(x) = b_j \text{ . So,}$$

$$I_E(s) = \sum_{i=1}^{M}\sum_{j=1}^{N} a_i \mu(C_{ij} \cap E)$$

$$\le \sum_{i=1}^{M}\sum_{j=1}^{N} b_j \mu(C_{ij} \cap E)$$

$$= I_E(t)$$

4. By monotonicity of μ we have,

$$I_F(s) = \sum_{i=1}^{M} a_i \mu(A_i \cap F)$$

$$\le \sum_{i=1}^{M} a_i \mu(A_i \cap E)$$

$$= I_E(s)$$

5. We know that if we have $E_1 \subseteq E_2 \subseteq E_3 \ldots$ $k=1$

$\lim_{n\to\infty} \mu(E_k) = \mu(E)$. Thus,

$\subseteq$.... and $E = \square^{\infty}$

E_k then

$\lim_{k\to\infty} I_{E_k}(s)$

$= \lim_{k\to\infty} \sum_{i=1}^{M} a_i \mu(A_i \cap E_k)$

$= \sum_{i=1}^{M} a_i \lim_{k\to\infty} \mu(A_i \cap E_k)$

$= \sum_{i=1}^{M} a_i \mu(A_i \cap E)$

$= I_E(s)$

Definition: If $f : X \to R^+$ is a non negative F measurable function, $E \in F$, then the integral of f over E is

$$\int_E f d\mu = \sup\{I_E(s) : s \text{ is a simple } F\text{-measurable function, } 0 \le s \le f\}$$

But, if $E \neq X$ we need only that f is defined on some domain containing E.

Let $I(f, E)$ denote the set,

$\{I_E(s) : s$ is a simple F-measurable function, $0 \le s \le f\}$ So the integral equals sup $I(f, E)$.

Note: The integral exists for all nonnegative F measurable functions, though it might be infinite.

If $\int_E f d\mu = \infty$ we say that the integral is defined.

If $\int_E f d\mu < \infty$ we say that f is μ-integrable or summable on E.

Theorem 13.2: For a non negative, F measurable simple function t, we have

$$\int_E t\,d\mu = I_E(t).$$

Proof: Given any simple F measurable function, $0 \le s \le t$ we have $I_E(s) \le I_E(t)$ by Theorem 13.1.

So $I_E(t)$ is an upper bound for $I(t, E)$ for which $\int_E t\,d\mu$ is the least of all upper bounds.

Hence,

$$\int_E t\,d\mu \le I_E(t)$$

Also, $\int_E t\,d\mu \ge I_E(s)$ for all simple F measurable function, $0 \le s \le t$ and so is greater than $I_E(s)$ for any particular s, namely $s = t$. Hence, $\int_E t\,d\mu \ge I_E(t)$.

Example 13.2: If $f \equiv k$, i.e., a constant, then $\int_E f\,d\mu = I_E(f) = k\mu(E)$.

Theorem 13.3: Consider that all sets are in F and all functions are non negative and F measurable.

1. For all $c \ge 0$,

$$\int_E cf\,d\mu = c\int_E f\,d\mu$$

2. If $0 \le g \le h$ on E then,

$$\int_E g\,d\mu \le \int_E h\,d\mu$$

3. If $E_1 \subseteq E_2$ and $f \ge 0$ then,

$$\int_{E_1} f\,d\mu \le \int_{E_2} f\,d\mu$$

…(13.1)

Proof:

1. If $c = 0$ then both the right hand side and left hand side of Equation (13.1) are 0. Assume $c > 0$.

If $0 \le s \le cf$ is a simple F measurable function then so is $0 \le \frac{1}{c} s \le f$.
Thus,

c

$fd\mu \ge I$

$\left(\frac{1}{c} s\right) = \frac{1}{c} I$

(s)

$\int_E$ $_E$ c c $_E$

By Theorem 13.1 (1).

Hence, $c \int_E fd\mu$ is an upper bound for $I(cf, E)$ for which $\int_E cfd\mu$ is the least upper bound. Thus, $c \int_E fd\mu \ge \int_E cfd\mu$.

Starting with the observation that if $0 \le s \le f$ is a simple F measurable function then so is $0 \le cs \le cf$ we obtain,

$\int_E cfd\mu \ge I_E(cs)$

By the definition of $\int_E$

$= cI_E(s)$ By Theorem13.1(1).

$\frac{1}{c}$ Hence, $\int_E$

$(cf)d\mu$ is an upper bound for $I(f, E)$ for which $\int_E$

$fd\mu$ is the

least upper bound. Hence $\frac{1}{c}\int_E$

$(cf)d\mu \geq$

$\int_E fd\mu$, or, $\int_E (cf)d\mu \geq c\int_E$

$fd\mu$.

On combining both inequalities, we get the result.

2. Let $0 \leq s \leq g$ be a simple, F measurable function. Then, since $g \leq h$ we trivially have $0 \leq s \leq h$ in which case $I_E(s) \leq \int_E hd\mu$ by the definition of integral $\int_E$.

Thus, $\int_E hd\mu$ is an upper bound for $I(g, E)$. As in (1) we get

$\int_E hd\mu \geq \int_E gd\mu$.

3. Let $0 \leq s \leq f$ be a simple, F measurable function. Then,

$I_{E1}(s) \leq I_{E2}(s)$ By Theorem 4.31(3)

$\leq \int_{E_2} fd\mu$ By the definition of $\int_{E_2}$

So $\int_{E_2} fd\mu$ is an upper bound for $I(f, E_1)$ and so is greater than the least of all upper bounds. Hence, $\int_{E_2} fd\mu \geq \int_{E1} fd\mu$.

Lemma 1: Let $E \in F$, $f \geq 0$ is F measurable and $\int_E fd\mu < \infty$. Set, $A = \{x \in E :$
$f(x) = +\infty\}$. Then, $A \in F$ and $\mu(A) = 0$.

Proof: Since f is F measurable, therefore $f^{-1}(\{\infty\}) \in F$ and so $A = E \cap f^{-1}$ $(\{\infty\}) \in F$. Define,

$$s_n(x) = \begin{cases} n & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Since $A \in F$, we infer that s_n is an F measurable simple function.

Also, $s_n \le f$ and so

$n\mu(A) = I_E(s_n)$ by definition of I_E

$\le \int_E f d\mu$ by definition of $\int_E$

$< \infty$

by assumption Which is true for all $n \ge 1$ means that $\mu(A) = 0$.

Lemma 2: If f is F measurable and non negative on $E \in F$ and $\mu(E) = 0$, then

$\int_E f d\mu = 0$.

Proof: Let $0 \le s \le f$ be a simple, F measurable function. So, $s = \sum_{n=1}^{N} a_n X_{A_n}$ for some $a_n \geq 0, A_n \in F$. Then $I_E(s) = \sum_{n=}^{N}$

$a_n \mu(A_n$

$\cap E)$. But μ is monotone which means that $\mu(A_n \cap E) \leq \mu(E) = 0$ for all n and so $I_E(s) = 0$ for all such simple functions. Hence, $I(f, E) = \{0\}$ and so $\int_E f d\mu = \sup I(f, E) = 0$.

Lemma 3: If $g \geq 0$ and $\int_E g d\mu = 0$, then $\mu\{x \in E : g(x) > 0\} = 0$.

Proof: Let $A = \{x \in E : g(x) > 0\}$ and $A_n = \{x \in E : g(x) > 1/n\}$.

Then, the sets $A_n = ? \cap \{x : g(x) > 1/n\} \in Fb$ satisfy $A_1 \subseteq A_2 \subseteq A_3 \subseteq$ ...with

$? = \bigcup_{n=1}^{\infty} A_n$.

By Lemma 1, $\mu(A) = \lim_{n\to\infty} \mu(A_n)$. Using,

$$s_n(x) = \begin{cases} \frac{1}{n} & \text{if } x \in A_n \\ 0 & \text{otherwise} \end{cases}.$$

so $s_n \leq g$ on A_n we have,

$$\frac{1}{n}\mu(A_n) = I_{A_n}(s_n)$$

$$\leq \int_A g d\mu$$

by the definition of $\int_A$

n n

$\leq \int_E gd\mu$

By Theorem 4.33(3)

n

= 0 Byassumption

So $\mu(A_n) = 0$ for all n and hence $\mu(A) = 0$.

Definition: If a property P holds on all points in $E \setminus A$ for some set A with $\mu(A) = 0$ then P is said to hold almost everywhere (μ) on E. It is possible that P holds on some of the points of A or that the set of points on which P does not hold is non measurable. But, if μ is a complete measure, such as the Lebesgue-Stieltjesmeasure

μ_F, thenthesituationis simpler.Assumethatapropertv$P$ holdsalmosteverywhere (μ) on E. The definition says that the set of points, D say, on which P does not hold, can be covered by a set of measure zero, i.e., there exists $A : D \subseteq A$ and

$\mu(A) = 0$.

However if μ is complete then D will be measurable of measure zero.

Lemma 4: If $g \geq 0$ and $\int_E gd\mu = 0$ then $g = 0$ almost everywhere (μ) on E. **Theorem 13.4:** If $g, h: X \to R^+$ are F measurable functions and $g \leq h$ almost everywhere (μ) then, $\int_E gd\mu \leq \int_E hd\mu$.

Proof: By assumption there exists a set $D \subseteq E$, of measure zero, such that for all

$x \in$?/D we have $g(x) \leq h(x)$. Let $0 \leq s \leq g$ be a simple, F measurable function, writtenas

$s = \sum_{i=1}^{N} a_i X_{A_i}$, with $\bigcup_{i=1}^{N} A_i = E$

Define, a simple, F measurable function

$$s^*(x) = \begin{cases} s(x) & \text{if } x \notin D \\ 0 & \text{if } x \in D \end{cases}$$

$$= \sum_{i=1}^{N} a_i X_{A_i \cap D^C}$$

Then, for $x \in D^c$ we have $s^*(x) = s(x) \le g(x) \le h(x)$, while for $x \in D$ we have $s^*(x) = 0 \le h(x)$. Thus, $s^*(x) \le h(x)$ for all $x \in E$. Note that, $A_i = (A_i \cap D^c) \cup (A_i \cap D)$, a disjoint union in which case $\mu(A_i) = \mu(A_i \cap D^c) \cup \mu(A_i \cap D) = \mu(A_i)$. But $A_i \cap D \subseteq D$ and so $\mu(A_i \cap D) \le \mu(D) = 0$. Thus, $\mu(A_i) = \mu(A_i \cap D^c)$.

Hence,

$$I_E(s^*) = \sum_{i=1}^{N} a_i \mu(A_i \cap D^C)$$

$$= \sum_{i=1}^{N} a_i \mu(A_i)$$

$$= I_E(s)$$

So, $I_E(s) = I_E(s^*) \leq \int_E hd\mu$ by the definition of integral $\int_E$. Thus, $\int_E hd\mu$ is an upper bound for $I(g, E)$ while $\int_E gd\mu$ is the least of all upper bounds for $I(g, E)$. Hence, $\int_E hd\mu \geq \int_E gd\mu$.

Corollary: If $g, h: X \to R^+$ are F measurable with $g = h$ almost everywhere (μ)
on E then,

$$\int_E gd\mu = \int_E hd\mu .$$

Proof: By assumption there exists a set $D \subseteq E$ of measure zero such that for all $x \in$ ⍰D we have $g(x) = h(x)$. In particular, for these x we have $g(x) \leq h(x)$ and $h(x) \leq g(x)$. So $g \leq h$ almost everywhere (μ) on E and $h \leq g$ almost everywhere
(μ) on E. Hence, the result follows from two applications of Theorem 4.34.

So, a function may have its values changed on a set of measure zero without changing the value of its integral. Particularly, we may assume that a non negative integrablefunction is finitevalued.

13.3.1 Monotone Convergence Theorem

The monotone convergence theorem is anyof a number of related theorems proving the convergence of monotonic sequences (sequences that are increasing or decreasing) thatarealso bounded. Informally, thetheoremsstate that if a sequence is increasing and bounded above bya supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below byan infimum, it will converge to the infimum.

Theorem 13.5 Monotone Convergence: Let (f_n) be non decreasing sequence of non negative measurable functions with limit *f*.

Then,

$$\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu,$$

$A \in$ A

Proof: First, note that $f_n(x) \le f(x)$ so that

$$\lim_n \int_A f_n d\mu \le \int f d\mu$$

It is remained to prove the opposite inequality.

For this it is enough to show that for any simple φ such that $0 \le \varphi \le f$ the following inequalityholds,

$$\int_A \varphi d\mu \le \lim_n \int_A f_n d\mu$$

Take $0 < c < 1$. Define,

$$A_n = \{x \in A : f_n(x) \ge c\varphi(x)\}$$

Then $A_n \subset A_{n+1}$

and $A = \bigcup_{n=1}^{\infty} A_n$. Observe that,

$$c\int_A \varphi d\mu = \int_A c\varphi d\mu = \lim_{n\to\infty} \int_{A_n} c\varphi d\mu$$

$$\le \lim_{n\to\infty} \int_{A_n} f_n d\mu \le \lim_{n\to\infty} \int f_n d\mu$$

Pass to the limit $c \to 1$.

Theorem 13.6: Let $f = f + f$; f, $f \in L^1(\mu)$.

$1 \quad 2 \quad 1 \quad 2$

Then

$$\int f d\mu = \int f_1 d\mu + \int f_2 d\mu .$$

$f \in L^1(\mu)$

and

Proof: First, let $f, f \geq 0$. If theyare simple then the result is trivial.

Otherwise,

$1\ 2$

choosemonotonicallyincreasingsequences (φ

),(φ

) such that φ

$\to f$ and

φ $\to f$. Then for $\varphi = \varphi$

$+ \varphi$,

$n,1$

$n,2$

$n,1 \quad 1$

$n,2 \quad 2$

$n \quad n,1$

$n,2$

$$\int \varphi_n d\mu = \int \varphi_{n,1} d\mu + \int \varphi_{n,2} d\mu$$

and the result follows from Theorem 4.35.

If $f_1 \geq 0$ and $f_2 \leq 0$, put

$$A = \{x : f(x) \geq 0\},\ B = \{x : f(x) < 0\}$$

Then, f, f and $-f$ are non negative on A. $1 \quad 2$

Hence, $\int_A f_1 = \int_A f_A\ f d\mu + \int_A (-f_2) d\mu$.

Similarly,

$$\int_B (-f_2) d\mu = \int_B f_1 d\mu + \int_B (-f) d\mu$$

The result follows from the additivity of integral.

Theorem 13.7: Let $A \in \mathcal{A}$, (f_n) be a sequence of non negative measurable functions and

$$f(x) = \sum_{n=1}^{\infty} f_n(x),$$

$x \in A$

then,

$$\int_A f d\mu = \sum_{n=1}^{\infty} \int_A f_n d\mu .$$

13.3.2 Fatou's Lemma

Fatou's lemma establishes an inequality relating the Lebesgue integral of the limit inferior of a sequence of functions to the limit inferior of integrals of these functions. The lemma is named after Pierre Fatou. Fatou's lemma can be used to prove the Fatou–Lebesgue theorem and Lebesgue's dominated convergence theorem.

Theorem 13.8 (Fatou's Lemma): If (f_n) is a sequence of non negative measurable functions defined almost everywhere and $f(x) = \lim_{n\to\infty} f_n(x)$, then

$$\int_A f d\mu \le \lim_{n\to\infty} \int_A f_n d\mu \text{ where } A \in \mathcal{A} .$$

Proof: Put $g_n(x) = \inf_{i \ge n} f_i(x)$.

Then, by definition of the lower limit $\lim_{n\to\infty} g_n(x) = f(x)$.

Moreover, $g_n \le g_{n+1}$, $g_n \le f_n$. By the monotone convergence theorem,

$$\int_A f d\mu = \lim_n \int_A g_n d\mu = \underline{\lim}_n \int_A g_n d\mu \leq \underline{\lim}_n \int_A f_n d\mu$$

Hence the theorem is proved.

13.4 GENERAL LEBESGUE INTEGRAL

Define the positive part f^+ and negative part f^- of a function as,

o

,

$$f = f^+ - f^-$$

$$|f| = f^+ + f^-$$

Definition: A measurable function f is said to be integrable over E if f^+ and f^-

are both integrable over E. In this case we define,

$$\int_E f = \int_E f^+ - \int_E f^-$$

Theorem 13.9: Let f and g be integrable over E. Then,

(i) The function $f + g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$.

(ii) If $f \leq g$ almost everywhere then, $\int_E f \leq \int_E g$.

(iii) If A and B are disjoint measurable sets contained in E, then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

Proof: From the definition, it follows that the functions f^+, f^-, g^+, g^- are all integrable. If $h = f + g$, then $h = (f^+ - f^-) + (g^+ - g^-)$ and hence $h = (f^+ + g^+) - (f^- + g^-)$. Since, $f^+ + g^+$ and $f^- + g^-$ are integrable therefore we then have,

— —

$$\int_E h = \int_E [(f^+ + g^+) - (\overline{f} + \overline{g})]$$

$$= \int_E (f^+ + g^+) - \int_E (\overline{f} + \overline{g})$$

$$= \int_E f^+ + \int_E g^+ - \int_E \overline{f} - \int_E \overline{g}$$

That is,

$$\int_E (f+g) = (\int_E f^+ - \int_E \overline{f}) + (\int_E g^+ - \int_E \overline{g})$$

$$= \int_E f + \int_E g$$

Proof of (*ii*) follows from part (*i*) and the fact that the integral of a non negativeintegrablefunction is nonnegative.

For the proof of (*iii*) we have,

$$\int_{A\cup B} f = \int_{A\cup B} f\chi = \int f\chi_A + \int f\chi_B$$

$$= \int_A f + \int_B f$$

Now, $f+g$ is not defined at points where $f=\infty$ and $g=-\infty$, and where $f=-\infty$ and $g=\infty$. However, the set of such points must have measure equal to 0,

since f and g are integrable. Hence, the integrability and thevalue of independent of the choice of values in these ambiguous cases.

$\int (f+g)$ is

Theorem 13.10: Let f be a measurable function over E. Then f in integrable over E iff $|f|$ is integrable over E. Furthermore, if f is integrable, then

$$\left|\int_E f\right| \le \int_E |f| .$$

Proof: If f is integrable then both f^+ and f^- are integrable. But $|f| = f^+ + f^-$. Hence, integrability of f^+ and f^- implies the integrability of $|f|$.

Moreover, if f is integrable, then since $f(x) \le |f(x)| = f(x)$, the property which states that if $f \le g$ almost everywhere then, $\int f \le \int g$ implies that

$$\int f \le \int |f| \quad \ldots(4.17)$$

On the other hand since $-f(x) \le |f(x)|$, we have

$$\int f \le \int |f| \ldots.(4.18)$$

From Equations (4.17) and (4.18) we have,

$$\left|\int f\right| \le \int |f|$$

Conversely, suppose f is measurable and suppose $|f|$ is integrable. Since, $0 \le f^+(x) \le |f(x)|$, it follows that f^+ is integrable. Similarly, f^- is also integrable and hence f is integrable.

Lemma: Let f be integrable. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|\int_A f| < \varepsilon$ whenever A is a measurable subset of E with $mA < \delta$.

Proof: When f is non negative, the lemma is proved. Now for arbitrary measurable function f we have $f = f^+ - f^-$. So, given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that,

$$\int_A f^+ < \frac{\varepsilon}{2}$$

when $mA < \delta_1$. Similarly there exists $\delta_2 > 0$ such that

$$\int_A f^- < \frac{\varepsilon}{2}$$

when $mA < \delta_2$. Thus, when $mA < \delta = \min(\delta_1, \delta_2)$, we have

$$\left|\int_A f\right| \le \int_A |f| = \int_A f^+ + \int_A f^- < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, the lemma is proved.

13.4.1 Lebesgue Convergence Theorem

Theorem 13.11 (Lebesgue's dominated convergence theorem): Let $A \in$ A ,

(f) be a sequence of measurable functions such that $f(x) \to f(x)$ $(x \in A)$. If there

n n

exists a function $g \in L^1(\mu)$ on A such that,

$$|f_n(x)| \leq g(x)$$

then,

$$\lim_n \int_A f_n d\mu = \int_A f d\mu .$$

Proof: From $|f_n(x)| \leq g(x)$ we get f_n n

Fatou's lemma it follows that,

$$\int_A (f + g) d\mu \leq \underline{\lim}_n \int_A (f_n + g)$$

or,

$$\int_A f d\mu \leq \underline{\lim}_n \int_A f_n d\mu$$

$\in L^1(\mu)$. As f

$+ g \geq 0$ and $f + g \geq 0$, by

Since $g - f_n \geq 0$, in the same way

$$\int_A (g - f) d\mu \leq \underline{\lim}_n \int_A (g - f_n) d\mu$$

So that,

$$-\int_A f d\mu \leq -\underline{\lim}_n \int_A f_n d\mu$$

which is the same as

$$\int_A f d\mu \geq \overline{\lim}_n \int_A f_n d\mu$$

Hence,

$$\underline{\lim}_n \int_A f_n d\mu = \overline{\lim}_n \int_A f_n d\mu = \int_A f d\mu$$

Check Your Progress

1. Define integral of non negative function.
2. State monotone convergence theorem.
3. Give the statement of Fatou's lemma.
4. Write the condition for a measurable function to be integrable.
5. State Lebesgue's dominated convergence theorem.

13.5 LET US SUM UP

- Let s be a non negative F measurable simple function so that, $s = \sum_{i=1}^{N} a_i X_{A_i}$ with disjoint F measurable sets A_i, $\cup_{i=1}^{N} A_i = X$ and $a_i \geq 0$.
- For any $E \in F$ define the integral of f over E to be, $I_E(s) = \sum_{i=1}^{N} a_i \mu(A_i \cap E)$ with the convention that if $a_i = 0$ and $m(A_i \cap E) = +\infty$ then $0 \times (+\infty) = 0$. So the area under $s \equiv 0$ in R is zero.
- If $E_1 \subseteq E_2 \subseteq E_3$ … are in F and $E = \cup_{n=1}^{\infty} E_n$ then, $\lim_{n\to\infty} \mu(E_n) = \mu(E)$ and we say that we have an increasing sequence of sets.
- If there exists an n such that $\mu(E_n) = +\infty$ then $E_n \subseteq E$ implies $\mu(E) = +\infty$ and the result follows.

- If $f : X \rightarrow R^{+}$ is a non negative F measurable function, $E \in F$, then the integral of f over E is

  $$\int_E f d\mu = \sup\{I_E(s) : s \text{ is a simple } F\text{-measurable function, } 0 \leq s \leq f\}$$

  But, if $E \neq X$ we need only that f is defined on some domain containing E.

- Let $I(f, E)$ denote the set,

  $\{I_E(s) : s$ is a simple F-measurable function, $0 \leq s \leq f\}$

  So the integral equals sup $I(f, E)$.

- For a nonnegative, F measurable simple function t, we have $\int_E t d\mu = I_E(t)$.

- So $I_E(t)$ is an upper bound for $I(t, E)$ for which $\int_E t d\mu$ is the least of all upper bounds.

- If $c = 0$ then both the right hand side and left hand side of Equation (13.1) are 0. Assume $c > 0$.

- Starting with the observation that if $0 \leq s \leq f$ is a simple F measurable function then so is $0 \leq cs \leq cf$ we obtain,

  $$\int_E cf d\mu \geq I_E(cs)$$

By the definition of $\int_E = cI_E(s)$

$\frac{1}{}$ ( )

- Hence,

$c\int_E cf$

$d\mu$ is an upper bound for $I(f, E)$ for which

$\int_E f d\mu$ is the

least upper bound. Hence

$c\int_E f d\mu$.

$c\int_E$

$(cf)d\mu \geq$

$\frac{1}{}$

$\int_E f d\mu$, or, $\int_E (cf) d\mu \geq$

- D^c). Let $0 \leq s \leq g$ be a simple, F measurable function. Then, since $g \leq h$ we trivially have $0 \leq s \leq h$ in which case $I_E(s) \leq \int_E h d\mu$ by the definition of integral $\int_E$.

- Let $E \in F$, $f \geq 0$ is F measurable and $\int_E f d\mu < \infty$. Set, $A = \{x \in E : f(x) = +\infty\}$. Then, $A \in F$ and $\mu(A) = 0$.

- If f is F measurable and non negative on $E \in F$ and $\mu(E) = 0$, then $\int_E f d\mu = 0$.

- Let $A = \{x \in E : g(x) > 0\}_n$ and $A = \{x \in E : g(x) >$

$1/n\}$.

Then, the sets $A_n = ⍰ \cap \{x : g(x) > 1/n\} \in Fb$ satisfy $A_1 \subseteq A_2 \subseteq A_3 \subseteq ...$ with

$⍰ = \bigcup_{n=1}^{\infty} A_n$.

- If a property P holds on all points in $E \setminus A$ for some set A with $\mu(A) = 0$ then P is said to hold almost everywhere (μ) on E. It is possible that P holds on some of the points of A or that the set of points on which P does not hold is non measurable. But, if μ is a complete measure, such as the Lebesgue- Stieltjesmeasure μ_F, then thesituation is simpler.

- Assume that a property P holds almost everywhere (μ) on E. The definition says that the set of points, D say, on which P does not hold, can be covered by a set of measure zero, i.e., there exists $A : D \subseteq A$ and $\mu(A) = 0$.

- By assumption there exists a set $D \subseteq E$, of measure zero, such that for all $x \in ⍰/D$ we have $g(x) \le h(x)$. Let $0 \le s \le g$ be a simple, F measurable function, written as

$$s = \sum_{i=1}^{N} a_i X_{A_i}, \text{ with } \bigcup_{i=1}^{N} A_i = E$$

- Then, for $x \in ⍰D$ we have $s^*(x) = s(x) \le g(x) \le h(x)$, while for $x \in D$ we have $s^*(x) = 0 \le h(x)$. Thus, $s^*(x) \le h(x)$ for all $x \in E$. Note that, $A_i = (A_i \cap D^c) \cup (A_i \cap D)$, a disjoint union in which case $\mu(A_i) = \mu(A_i \cap D^c) \cup \mu(A_i \cap D) = \mu(A_i)$. But $A_i \cap D \subseteq D$ and so $\mu(A_i \cap D) \le \mu(D) = 0$. Thus, $\mu(A_i) = \mu(A_i \cap ⍰_i$

- A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In this case we define,

$$\int_E f = \int_E f^+ - \int_E f^-$$

- The function $f+g$ is integrable over E and $\int_E (f+g) = \int_E f + \int_E g$.

- If $f \leq g$ almost everywhere then, $\int_E f \leq \int_E g$.

- If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f.$$

13.6 KEY WORDS

Monotone convergence theorem: The monotone convergence theorem is any of a number of related theorems proving the convergence of monotonic sequences that are also bounded. Informally, the theorems state that if a sequence is increasing and bounded above by a supremum, then the sequence will converge to the supremum; in the same way, if a sequence is decreasing and is bounded below by an infimum, it will converge to the infimum.

Fatou's lemma: Fatou's lemma establishes an inequality relating the Lebesgue integral of the limit inferior of a sequence of functions to the limit inferior of integrals of these functions. The lemma is named after Pierre Fatou.

13.7 QUESTIONS FOR REVIEW

1. Define integral of nonnegative functions.
2. Where is monotone convergence theorem applied?

3. State Fatou's lemma with an example.
4. State general Lebesgue integral.
5. Write an application of Lebesgue convergence theorem.
6. Explain integral of nonnegative functions with examples.

State and prove monotone convergence theorem 8.Explain Fatou's lemma with the help of examples.

9.Discuss between general Lebesgue integral and Lebesgue convergence theorem.

13.8 SUGGESTED READINGS AND REFERENCES

Rudin, Walter. 1976. *Principles of Mathematics Analysis,* 3rd edition. New York: McGraw Hill.

Carothers, N. L. 2000. *Real Analysis,* 1st edition. UK: Cambridge University Press.

Rudin, Walter. 1986. *Real and Complex Analysis,* 3rd edition. London: McGraw- Hill Education– Europe.

Barra, G. De. 1987. *Measure Theory and Integration*. New Delhi: Wiley Eastern Ltd.

Royden, H. L. 1988. *Real Analysis,* 3rd edition. New York: Macmillan Publishing Company.

Malik, S. C. and Savita Arora. 1991. *Mathematical Analysis*. New Delhi: Wiley Eastern Limited.

13.9 ANSWERS TO CHECK YOUR PROGRESS

1. Let s be a non negative F measurable simple

zfunction so that, $s = \sum a_i X_{A_i}$ N

with disjoint F measurable sets A_i, $\bigcup_{i=1}^{N} A_i = X$ and $a_i \geq 0$. For any $E \in F$ define the integral of f over E to be, $I_E(s) = \sum_{i=1}^{N} a_i \mu(A_i \cap E)$ with the convention that if $a_i = 0$ and $\mu(A_i \cap E) = +\infty$ then $0 \times (+\infty) = 0$. So the area under $s \equiv 0$ in R is zero.

2. Let (f_n) be non decreasing sequence of non negative measurable functions with limit f.

Then, $\int_A f d\mu = \lim_{n\to\infty} \int_A f_n d\mu$,

$A \in$ A

3. If (f_n) is a sequence of non negative measurable functions defined almost

everywhere and $f(x) = \lim_{n\to\infty}$

$A \in$ A .

$f_n(x)$, then $\int_A f d\mu \leq \lim_{n\to\infty} \int_A f_n d\mu$ where

4. A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In this case we define,

$$\int_E f = \int_E f^+ - \int_E f^-$$

5. Let $A \in$ A , (f_n) be a sequence of measurable functions such that $f_n(x) \to f(x)$ $(x \in A)$. If there exists a function $g \in L^1(\mu)$ on A such that,

$$|f_n(x)| \leq g(x)$$

then, $\lim_{n} \int_A f_n \, d\mu = \int_A f \, d\mu$.

UNIT 14 LEBESGUE INTEGRAL: RIEMANN INTEGRAL

STRUCTURE

14.1 OBJECTIVES

After going through this unit, you will be able to:

- Discusstheshortcomings ofRiemannintegral
- InterpretLebesgueintegralofaboundedfunctionoverasetoffinitemeasure
- KnowLebesgueintegralas a generalizationof Riemannintegral

14.2 INTRODUCTION

Riemann integration is the formulation of integration. Many other forms of integration, notably Lebesgue integrals, are extensions of Riemann integrals to larger classes of functions. The Riemann integral was developed by Bernhard Riemann in 1854 and was, when invented, the first rigorous definition of integration applicable to not necessarily continuous functions.

The integral now has more significancethanthe anti-operation of the derivative. There are now multiple integrals with increasingly greater range of use, yet Riemann integration is sufficient for nearlyallphysicalproblems.

In this unit, you will study about the shortcomings of Riemann integral, Lebesgueintegral of aboundedfunctionoveraset of finitemeasureand Lebesgue integral as a generalizationofRiemannintegral in detail.

14.3 LEBESGUE INTEGRAL: RIEMANN INTEGRAL

WhiletheRiemannintegralissufficientinmostdailysituations,itfallsshorttomeet our needs in quite a lot of importantways. First, the class of Riemannintegrable functions is relatively small. Second, the Riemann integral does not have satisfactory limit properties. That is, given a sequence of Riemann integrable functions $\{f_n\}$ witha limit function $f = \lim_{n\to\infty} f_n$, it does not necessarilyfollowthatthelimit function f is Riemann integrable. Third, all L_p spaces except for L_∞ fail to be complete under the Riemann integral.

Example 12.1: Consider the sequence of functions $\{f_n\}$ over the interval $E = [0, 1]$.

$$f_n(x) = \begin{cases} 2^n & \text{if } \dfrac{1}{2^n} \le x \le \dfrac{1}{2^{n-1}} \\ 0 & \text{otherwise} \end{cases}$$

Thelimitfunctionofthis sequenceissimply$f=0$. Inthis example, each function in the sequence is integrable as is the limit function. However, the limit of the sequence of integrals is not equal to the integral of the limit of the sequence. Thatis,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = 1 \neq 0 = \int_0^1 \lim_{n\to\infty} f_n(x)\, dx$$

Example 12.2: Consider the sequence of functions $\{d_n\}$ over the interval $E = [0, 1]$.

$$d_n(x) = \begin{cases} 1 & \text{if } x \in \{r_n\} \\ 0 & \text{otherwise} \end{cases}$$

where $\{r_n\}$ is the set of the first n elements of some decided upon enumeration of the rational numbers. Each function d_n is Riemann integrable since, it is discontinuous only at n points. Thelimit function $D = \lim_{n\to\infty} d_n$ is given by

$$D(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ isirrational} \end{cases}$$

This function, knows as the Dirichlet function, is discontinuous everywhere and therefore not Riemann integrable. Another way of showing that $D(x)$ is not Riemann integrable is to take upper and lower sums, which result in 1 and 0, respectively.

14.3.1 Lebesgue Integral of a Bounded Function over a Set of Finite Measure and its Properties

The Lebesgue Integral of a Bounded Function

Younow know some of theshortcomingsofthe Riemannintegral. In particular, we would like a function, which is 1 on a measurable set and 0 elsewhere, to be integrable and haveits integral the measure of the set.

The function χ_E defined by,

$$\chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E \end{cases}$$

is called the characteristic function on E. A linear combination,

$\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is known as a simple function if the sets E_i are measurable.

This is not a unique representation for f. However, we note that a function f is simple iff it is measurable and assumes only a finite number of values. If f is a simple function and $[a_1, \ldots, a_n]$ are the set of non zero values of f, then $\phi = \sum a_i \chi_{A_i}$, where $A_i = \{x \mid \phi(x) = a_i\}$. This representation for ϕ is known as the canonical representation and is characterized by the fact that the A_i's are disjoint and the a_i distinct and non zero. If ϕ vanishes outside a set of finite measure, we define the integral of ϕ by

$\int \phi(x)dx = \sum_{i=1}^{n} a_i m A_i$ when ϕ has the canonical representation $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$. We

$i=1$

$i=1$

usuallyreduce the expression for this integral to $\int\varphi$. If E is anymeasurable

set, we define $\int\phi = \int\phi \cdot \chi_E$.

E

Lemma: If $E_1, E_2, \ldots, E_n$ are disjoint measurable subsets of E then every linear

n

combination $\phi = \sum c_i \chi_{E_i}$ with real coefficients c , c , …, c is a simple function

$i=1$

1 2 n

and

n

$\int\phi = \sum c_i \, mE_i$.

$i=1$

Proof: It is clear that ϕ is a simple function. Let a , a , …, a denote the non zero

1 2 n

real number in $\phi(E)$. For each $j = 1, 2, \ldots, n$ let,

$A_j = \square \; E_i$

$c_i = a_j$

Then we have, $A = \phi^{-1} (a) = \{x \mid \phi(x) = a\}$ and the canonical representation

j j j

n

$\phi = \sum a_j \chi_{A_j}$ $j=1$

Consequently, we obtain

$$\int \phi = \sum_{j=1}^{n} a_j mA_j$$

$$= \sum_{j=1}^{n} a_j m \left[\bigcup_{c_i = a_j} E_i \right]$$

$$= \sum_{j=1}^{n} a_j \sum_{c_i = a_j} mE_i$$

(Additivity of measures applies, since E_i aredisjoint)

$$= \sum_{j=1}^{n} c_j mE_i$$

Hence, the theorem is proved.

Theorem 12.1: Let ϕ and ψ be simple functions which vanish outside a set of finitemeasure. Then

$\int (a\phi + b\psi) = a\int\phi + b\int\psi$, and, if $\phi \geq \psi$ almost everywhere, then $\int\varphi \geq \int\psi$

.

Proof: Suppose $\{A_i\}$ and $\{B_i\}$ are the sets that occur in the canonical representations of ϕ and ψ. Let A_0 and B_0 be the sets where ϕ and ψ are zero.

Thenthesets E obtained bytaking all theintersections $A \cap B$ forma finite disjoint

$k \qquad i \quad j$

collection of measurable sets, and we have

$$\phi = \sum_{k=1}^{N} a_k \chi_{E_k}$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k}$$

and hence

$$a\phi + b\psi = a \sum_{k=1}^{N} a_k \chi_{E_k} + b \sum_{k=1}^{N} b_k \chi_{E_k}$$

$$= \sum_{k=1}^{N} aa_k \chi_{E_k} + \sum_{k=1}^{N} bb_k \chi_{E_k}$$

So,

$$= \sum_{k=1}^{N} (aa_k + bb_k) \chi_{E_k}$$

$$(a\phi + b\psi) = \sum_{k=1}^{N}(aa_k + bb_k)mE_k$$

$$= \sum_{k=1}^{N}(aa_k)mE_k + \sum_{k=1}^{N}(bb_k)\, mE_k$$

$$= a\sum_{k=1}^{N} a_k\, mE_k + b\sum_{k=1}^{N} b_k mE_k$$

$$= a\int\phi + b\int\psi \,.$$

To prove the second statement, notice that

$$\int\phi - \int\psi = \int(\phi - \psi) \geq 0$$

since the integral of a simple function which is greater than or equal to zero almost everywhere is non negative by the definition of the integral.

Theorem 12.2: Let f be defined and bounded on a measurable set E with mE finite. For

$$\inf_{f\leq\psi} \int_E \psi\,(x)dx = \sup_{f\geq\psi} \int_E \phi(x)dx$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f be measurable.

Proof: Suppose that f is bounded by M and that f is measurable. Then the sets

$$E_k = \left\{ x \mid \frac{KM}{n} \geq f(x) > \frac{(K-1)M}{n} \right\}, -n \leq K \leq n$$

are measurable, disjoint and have union E. Thus,

$$\sum_{k=-n}^{n} mE_k = mE$$

Thesimplefunctiondefinedby,

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} k\chi_{E_k}(X)$$

and

$$\phi_n(X) = \frac{M}{n}$$

satisfy,

$$\sum_{k=-n}^{n}$$

$(k-1)\chi_{E_k}(X)$

$\phi_n(X) \le f(x) \le \psi_n(X)$

Thus,

M n

$\inf \int \psi(x)dx \le \int \psi_n(x)\, dx =$ n $\sum km\, E_k$

E E

and

$k=-n$

M n

$\sup \int \phi(x)dx \ge \int \phi_n(x)\, dx =$ $\overline{}$ n

? ?

whence,

$\sum$

$k=-n$

$(k-1)m\, E_k$

$0 \le \inf$

$\int \psi(x)\, dx - \sup$

$\phi(x)dx \le {}^{M}$

$\int$

n

$\sum m\, E_k = {}^{M}\, mE$

n

E $\overline{E}$ $\overline{n}$ $k=-n$

Since, n is arbitrary we have

inf

$\int \psi(x)dx - \sup \int \phi(x)dx = 0$

E E

and the condition is sufficient. Consider now that,

$\inf \int \psi(x)dx = \sup \int \phi(x)dx$

$\psi \geq f$ E

$\phi \leq f$ E

Then given n, there are simple functions ϕ and ψ

such that

n n

$\phi_n(x) \leq f(x) \leq \psi_n(x)$ and

$\int \psi_n(x)\,dx - \int \phi_n(x)dx < \frac{1}{n}$

Then, the functions

$\psi^* = \inf \psi$

n

….. (12.1)

and $\phi^* = \sup \phi$ are measurable and also $\phi^*(x) \leq f(x) \leq \psi^*(x)$.

n

Now, the set $\Delta = \{x \mid \phi^*(x) < \psi^*(x)\}$ is the union of the sets

$\Delta_v = \{x \mid \phi^*(x) < \psi^*$

1

$(x) - \frac{}{v}\}$.

—

But every Δ is contained in the set $\{x \mid \phi_n(x) < \psi_n(x) - \frac{1}{}\}$, and the set

v

v

Equation (12.1) has measure less than v/n. Since n is arbitrary, $m\Delta_v = 0$ and

so

$m\Delta = 0$. Thus $\phi^* = \psi^*$ except on a set of measure zero, and $\phi^* = f$ except on a set of measure zero. Thus f is measurable and the condition is also necessary.

Definition: If f is a bounded measurable function defined on a measurableset E with finite mE, then we define the Lebesgue integral of f over E by,

$$\int_E f(x)dx - \inf \int_E \psi(x)dx$$

for all simple functions $\psi \geq f$.

By Theorem 12.3, we can also define this as

$$\int_E f(x)dx = \sup \int_E \phi(x)dx$$

for all simple functions $\phi \leq f$.

In fact, we sometimeswrite the integral as $\int_E f$. Also, if $E = [a, b]$ we write

$$\int_a^b f$$ instead of

$$\int_{[a,b]} f.$$

Theorem 12.3: If f and g are bounded measurable functions defined on a set E of finitemeasure, then

$$\int_E af = a\int_E f.$$

(ii)

$$\int_E f+g = \int_E f + \int_E g.$$

(i) If $f \leq g$ almost everywhere then $\int_E f$

(i) If $f = g$ almost everywhere then $\int_E f$

$\le \int_E g$.

$= \int_E g$.

(ii) If $A \le f(x) \le B$, then $AmE \le \int_E f \le BmE$.

(iv) If A and B are disjoint measurable sets of finite measure, then

$$\int_{A \cup B} f = \int_A g + \int_B f.$$

Proof: We know that if ψ is simple function then so is $a\psi$. Hence,

$$\int_E af = \inf_{\psi \ge f} \int_E a\psi = a \inf_{\psi \ge f} \int_E \psi = a \int_E f$$

which proves (i)

For the proof of (ii) let ε denote any positive real number. There are simple functions $\phi \le f$, $\psi \ge f$, $\xi \le g$ and $\eta \ge g$ that satisfy

$$\int_E \phi(x)dx > \int_E f - \varepsilon, \qquad \int_E \psi(x)dx < \int_E f + \varepsilon,$$

$$\int_E \xi(x)dx > \int_E g - \varepsilon, \int_E \eta(x)dx < \int_E g + \varepsilon,$$

Since, $\phi + \xi \le f + g \le \psi + \eta$, we have

$$\int_E (f + g) \ge \int_E (\phi + \xi) = \int_E \phi + \int_E \xi > \int_E f + \int_E g - 2\varepsilon$$

$$\int_E (f+g) \le \int_E (\psi+\eta) = \int_E \psi + \int_E \eta < \int_E f + \int_E g + 2\varepsilon$$

Since these hold true for every $\varepsilon > 0$, we have

$$\int_E (f+g) = \int_E f + \int_E g$$

For the proof of (*iii*) it is sufficient to establish,

$$\int_E (g - f) \ge 0$$

For everysimple function $\psi \ge g - f$, we have $\psi \ge 0$ almost everywhere in E. This means that,

$$\int_E \psi \ge 0$$

Hence, we obtain

$$\int_E (g - f) = \inf_{\psi \ge (g-f)} \int_E \psi(x)dx \ge 0 \qquad \ldots(12.2)$$

which establishes (*iii*)

In the same way, we can show that

$$\int_E (g - f) = \sup_{\psi \le (g-f)} \int_E \psi(x)dx \le 0 \qquad \ldots(12.3)$$

Therefore, from Equations (12.2) and (12.3) the result (*iv*) follows. In order to prove (*v*) we are given that,

$A \le f(x) \le B$

Apply (iv) to get,

$$\int_E f(x)dx \le \int_E Bdx = B\int_E dx$$

$$= BmE$$

That is,

$$\int_E f \le BmE$$

Similarly, we can prove that $\int_E f \ge AmE$

Now, we prove (*vi*). Recall that,

$\chi_{A \cup B} = \chi_A + \chi_B$

Therefore,

$$\int_{A\cup B} f = \int_{A\cup B} \chi_{A\cup B} f = \int_{A\cup B} f(\chi_A + \chi_B)$$

$$= \int_{A\cup B} f\chi_A + \int_{A\cup B} f\chi_B$$

$$= \int_A f + \int_B f$$

which proves the theorem.

14.3.2 Lebesgue Integral as A Generalization of Riemann Integral

Any function which is Riemann integrable is Lebesgue integrable as well and positively with the same values for the two integrals. Let us prove this formally. First, we recall one definition of Riemannintegrability. This definition is different from most, but is easilyseen to be equivalent; it makes our proofs a good deal simpler. Let $f : A \rightarrow R$ be a bounded function on a bounded rectangle $A \subseteq R^m$. Consider R-valued functions that are simplewith respect to a rectangular partition of A, otherwise known as step functions. Step functions are obviouslybothRiemann and Lebesgue integrable with the same values for the integral. The lower and upper Riemann integrals for f are,

$$L(f) = \sup\left\{\int_A l\, d\lambda : \text{step function } l \leq f\right\}$$

$$U(f) = \inf\left\{\int_A u\, d\lambda : \text{step function } u \geq f\right\}$$

We always have $L(f) \leq U(f)$; we say that f is Riemann integrable if $L(f) = U(f)$, andtheRiemannintegralof$f$isdefinedas $L(f) = U(f)$

Equivalently, f is Riemannintegrable when there exists sequence of lower simple functions $l_n \leq f$ and upper simple functions $u_n \geq f$, such that

$$\lim_{n\to\infty} \int_A l_n = L(f) = U(f) = \lim_{n\to\infty} \int_A u_n .$$

Theorem 12.4 (Riemann Integrability Implies Lebesgue Integrability): Let $A \subset R^m$ be a bounded rectangle. If $f : A \rightarrow R$ is properlyRiemann integrable, then it is also Lebesgue integrable with respect to Lebesgue measure with the same valuefortheintegral.

Proof: Pick a sequence l_n and u_n as above. Let $L = \sup_n l_n$ and $U = \inf_n u_n$.

Clearly, these are measurable functions, and we have $l_n \leq L \leq f \leq U \leq u_n$ Taking Lebesgue integral and takinglimits,

$$\lim_{n\to\infty} \int l_n \leq \int L \leq \int U \leq$$

lim

$n \to \infty$

$\int u_n$

Here, the limits on the two sides are the same, because the Riemann and

Lebesgue integrals for l_n and u_n coincide. So $\int (U - L) = 0$. Then $U = L$ almost everywhere, and U or L equals f almost everywhere. Since Lebesgue measure is complete, f is a Lebesgue measurable function.

Finally,the Lebesgueintegral $\int f$, which we now know exists, is squeezed inbetweenthetwolimits on the left andtheright, thatbothequalthe Riemann integral of f.

Check Your Progress

1. List the shortcomings of Riemann integral.
2. Define Lebesgue integral of f over measurable set E.
3. What are upper and lower Riemann integrals for f ?
4. State Lebesgue bounded convergence theorem.
5. State Lebesgue's criterion for integrability.

14.4 SUMMARY

- Anyfunctionwhich is Riemannintegrable is Lebesgueintegrable as well.
- Let $<f_n>$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $<f_n>$ is uniformly bounded, that is, there exists a real number M such that $| f_n(x) | \le M$, for all $n \in N$ and all $x \in E$. If

$\lim_{n\to\infty} f_n(x) = f(x)$ for each x in E, then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

- Let $f: [a, b] \to R$. Then, f is Riemannintegrableiff f is boundedandthe set of discontinuities of f has measure0.

- Let s be a non negative F measurable simple function so that $s = \sum_{i=1}^{N} a_i X_{A_i}$ with disjoint F measurable sets A_i, $\cup_{i=1}^{N} A_i = X$ and $a_i \geq 0$.

For any $E \in F$, we define the integral of f over E to be,

$I_E(s) = \sum_{i=1}^{N} a_i \mu(A_i \cap E)$ with the convention that if a_i

$\mu(A_i \cap E) = +\infty$ then $0 \times (+\infty) = 0$.

= 0 and

- Let (f_n) be non decreasing sequence of non negative measurable functions with limit f. Then $\int_A f d\mu = \lim_{n\to} \int_A f_n d\mu$,

$A \in \mathrm{A}$.

- If (f_n) is a sequence of non negative measurable functions defined almost everywhere and $f(x) = \lim_{n\to\infty}$ where $A \in \mathrm{A}$.

$f_n(x)$, then

$\int_A f d\mu \leq \lim_{n\to\infty} \int_A f_n d\mu$

- A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E.

- Let $A \in \mathrm{A}$, (f_n) be a sequence of measurable functions such that $f_n(x) \to$

$f(x)$ $(x \in A)$. If there exists a function $g \in L^1(\mu)$ on A such that, $|f_n(x)| \leq g(x)$, then $\lim_{n} \int_A f_n d\mu = \int_A f d\mu$.

14.5 KEYWORDS

- **Lebesgue integral:** The integral of a non-negative function of a single variable can be regarded, in the simplest case, as the area between the graph of that function and the x-axis.

- **Bounded function:** Afunction *f* defined on some set *X* with real or complex values is called bounded, if the set of its values is bounded.

14.6 QUESTIONS FOR REVIEW

1. What is Lebesgueintegral?
2. Briefa note on Riemannintegral.
3. Whataretheshortcomings of Riemannintegral?
4. Givethe properties of Lebesgueintegral of a boundedfunctionovera set of finitemeasure.
5. Define Lebesgueintegral as generalization of Riemannintegral.
6. Describeshortcomings of Riemannintegralusingillustrations.
7. Illustrate Lebesgueintegral of aboundedfunctionovera set of finite measure and itsproperties.
8. Discuss Lebesgueintegral as generalization of Riemannintegral.Prove that Reimannintegrabilityimplies Lebesgueintegrabilitywiththehelp of a theorem.

14.7 SUGGESTED READINGS AND REFERENCES

Rudin, Walter. 1976. *Principles of Mathematics Analysis,* 3rd edition. New York: McGraw Hill.

Carothers, N. L. 2000. *Real Analysis,* 1st edition. UK: Cambridge University Press.

Rudin, Walter. 1986. *Real and Complex Analysis,* 3rd edition. London: McGraw- Hill Education– Europe.

Barra, G.De. 1987. *Measure Theory and Integration.* New Delhi: WileyEastern Ltd.

Royden, H. L. 1988. *Real Analysis,* 3rdedition. NewYork: Macmillan Publishing Company.

Malik, S. C. and Savita Arora. 1991. *Mathematical Analysis*. New Delhi: Wiley Eastern Limited.
Gupta, S. L. and Nisha Rani. 2003. *Fundamental Real Analysis,* 4th edition.
New Delhi: Vikas Publishing House Pvt. Ltd.

14.8 ANSWERS TO CHECK YOUR PROGRESS QUESTIONS

1. While the Riemann integral is sufficient in most daily situations, it falls short to meet our needs in quite a lot of important ways. First, the class of Riemann integrable functions is relatively small. Second, the Riemann integral does not have satisfactory limit properties. That is, given a sequence of Riemann integrable functions $\{f_n\}$ with a limit function $f = \lim_{n\to\infty} f_n$, it does not necessarily follow that the limit function f is Riemann integrable. Third, all L_p spaces except for $L\infty$ fail to be complete under the Riemann integral.

2. If f is a bounded measurable function defined on a measurable set E with finite mE, then we define the Lebesgue integral of f over E by,

$$\int_E f(x)dx - \inf \int_E \psi(x)dx \text{ for all simple functions } \psi \geq f.$$

3. The lower and upper Riemann integrals for f are,

$$L(f) = \sup\left\{\int_A l\, d\lambda : \text{step function } l \leq f\right\}$$

$$U(f) = \inf\left\{\int_A u\, d\lambda : \text{step function } u \geq f\right\}$$

4. Let $<f_n>$ be a sequence of measurable functions

definedon a set E of finite measureandsuppose that$<f>$ is uniformlybounded, that is, there exists a real number M such that $|f(x)| \leq M$, for all $n \in N$ and all $x \in E$.

If $\lim_{n\to\infty} f_n(x) = f(x)$ for each X in E, then $\int_E f = \lim_{n\to\infty} \int_E f_n$.

5. Let $f : [a, b] \to R$. Then, f is Riemann integrable if and onlyif f is bounded and the set of discontinuities of f has measure 0.

14.9 SELF ASSESSMENT QUESTIONS AND EXERCISES

Short Answer Questions

9. What is Lebesgueintegral?
10. Briefa noteon Riemannintegral.
11. Whataretheshortcomings of Riemannintegral?
12. Givethe properties of Lebesgueintegral of a boundedfunctionovera set of finitemeasure.
13. DefineLebesgueintegral as generalizationofRiemannintegral.

Long Answer Questions

1. Describeshortcomings of Riemannintegralusingillustrations.
2. Illustrate Lebesgueintegral of aboundedfunctionovera set of finite measure and itsproperties.
3. Discuss Lebesgueintegral as generalization of Riemannintegral.
4. Provethat Reimannintegrabilityimplies Lebesgueintegrabilitywiththe help of atheorem.

14.10 FURTHER READINGS

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Carothers, N. L. 2000. *Real Analysis*, 1st edition. UK: Cambridge University Press.

Rudin, Walter. 1986. *Real and Complex Analysis*, 3rd edition. London: McGraw- Hill Education– Europe.

Barra, G.De. 1987. *Measure Theory and Integration*. New Delhi: WileyEastern Ltd.

Royden, H. L. 1988. *Real Analysis*, 3rdedition. NewYork: Macmillan Publishing Company.

Malik, S. C. and Savita Arora. 1991. *Mathematical Analysis*. New Delhi: Wiley Eastern Limited.

Gupta, S. L. and Nisha Rani. 2003. *Fundamental Real Analysis*, 4th edition.
New Delhi: Vikas Publishing House Pvt. Ltd.